

Computational Methods in Applied Sciences

Manolis Papadrakakis
George Stefanou
Vissarion Papadopoulos
Editors

Computational Methods in Stochastic Dynamics



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Preface

At the dawn of the twenty-first century, the considerable influence of inherent uncertainties on structural behavior has led the engineering community to recognize the importance of a stochastic approach to structural problems. Issues related to uncertainty quantification and its influence on the reliability of the computational models, are continuously gaining in significance. In particular, the problems of dynamic response analysis and reliability assessment of structures with uncertain system and excitation parameters have been the subject of extensive research over the last two decades as a result of the increasing availability of powerful computing resources and technology. This book focuses on advanced computational methods and software tools which can highly assist in tackling complex problems in stochastic dynamic/seismic analysis and design of structures. The selected chapters are authored by some of the most active scholars in their respective areas and represent some of the most recent developments in this field.

This edited book is primarily intended for researchers and post-graduate students who are familiar with the fundamentals and wish to study or to advance the state of the art on a particular topic in the field of computational stochastic structural dynamics. Nevertheless, practicing engineers could benefit as well from it as most code provisions tend to incorporate probabilistic concepts in the analysis and design of structures. The book consists of 16 chapters which are extended versions of papers presented at the recent COMPDYN 2009 and SEECCM 2009 Conferences. The chapters can be grouped into several thematic topics including dynamic response variability and reliability of stochastic systems, risk assessment, stochastic simulation of earthquake ground motions, efficient solvers for the analysis of stochastic systems, dynamic stability and stochastic modeling of heterogeneous materials.

In Chapter 1, G.I. Schuëller gives an overview of the well established deterministic model reduction techniques (Guyan reduction, component mode synthesis) and approaches for efficient uncertainty propagation in structural dynamics. The recent advances in the combination of these two fields are reviewed and methodologies for the consideration of uncertainty when using model reduction schemes are presented. The capabilities of a Karhunen-Loève expansion of the substructure matrices and of a random combination of substructures are investigated. It is shown that the proposed approaches can combine high accuracy with a substantial reduction of the computational cost compared to Monte Carlo simulation (MCS) of the full model.

In Chapter 2, S. Krenk and J. Høgsberg present a design principle for resonant control of a SDOF system by second order filters. The basic idea is the introduction of a resonant force with frequency tuning that results in splitting the original resonant mode into two modes with equal damping ratio. The design procedure is developed for resonant displacement and acceleration feedback, respectively, based on a combination of “equal modal damping” and approximately equal response amplitudes of the two modes. The procedure is extended to collocated resonant control of the lower modes of MDOF systems. An explicit design approach for the control parameters of MDOF systems is developed that includes the effect of the higher modes via a quasi-static approximation. The efficiency of resonant damping is illustrated by application to a benchmark example for stochastic wind load on a high-rise building.

S.K. Au presents a novel approach for improving importance sampling in the estimation of the first passage probability of nonlinear hysteretic systems subjected to stochastic earthquake excitations (Chapter 3). Instead of using fixed design point excitations, the importance sampling distribution is constructed using a stochastic process, called “adapted process”. A stochastic control algorithm is developed for the adapted process where the objective function reflects the expected energy needed to next yield as well as the exit velocity. The variance reduction efficiency of the “optimal” control law is investigated through a numerical example involving a SDOF elasto-plastic structure. It is shown that the proposed approach leads to substantial reduction of the “unit COV” of the importance sampling estimator while maintaining the required computational effort at a reasonable level (comparable to that of Markov Chain Monte Carlo-based approaches).

Another approach to the solution of the first passage problem for earthquake engineering applications is provided by M. Barbato in Chapter 4. This chapter describes the representation of the complex envelope process for transient response of linear structures and illustrates the use by the first passage problem. The Vanmarcke and modified Vanmarcke approximate solutions to the first passage problem are compared with the classical Poisson approximation and MCS results for an idealized linear elastic model of a steel building. The failure condition is expressed in terms of inter-storey drifts outcrossing specified deterministic thresholds. The retrofit of the benchmark structure with viscous dampers is also considered, allowing to illustrate the use of the closed-form approximations of the failure probability for non-classically damped linear elastic systems. The results presented show that the two Vanmarcke approximations can improve considerably the estimates of the failure probability for the first passage problem when compared with the simpler Poisson approximation.

A computationally efficient method for the stochastic analysis of large linear structural systems subjected to seismic actions is developed by P. Cacciola and G. Muscolino in Chapter 5. The method is based on modal analysis for the evaluation of stochastic response along with a modal correction approach, which includes the contribution of the neglected modes. Furthermore, a previously developed method for the evaluation of spectrum-compatible evolutionary power spectra of ground motions is presented. Some numerical results show the accuracy and

efficiency of the proposed technique for determining the spectral moments of the response.

In Chapter 6, S.M. Elachachi et al. investigate the effect of soil spatial variability (correlation length, coefficient of variation) on the reliability of buried networks subjected to earthquake loading. The reliability analysis is performed using a Response Surface Model (RSM) and the reliability index is calculated for serviceability and ultimate limit states. The dynamic response of a buried pipe subjected to natural ground motion records is computed taking into account a longitudinal variability of the properties of the soil. It is shown that the magnitude of the induced stresses and thus the reliability of the pipe are mainly affected by four parameters: a soil-structure length ratio, a soil-structure stiffness ratio, a structure-joint stiffness ratio (relative flexibility) and the magnitude of soil variability (coefficient of variation).

The problem of reliability-based optimization of structural systems under stochastic loading is treated by H. Jensen et al. in Chapter 7. In the proposed approach, a standard gradient-based algorithm with line search is used, allowing to explore the space of the design variables most efficiently. Subset simulation is adopted for the purpose of estimating the corresponding failure probabilities. The gradients of the failure probability functions are estimated by an approach based on the local behavior of the performance functions that define the failure domains. Numerical results show that only a moderate number of reliability estimates has to be performed during the entire design process. A numerical example with deterministic system properties is provided to show the effectiveness of the proposed approach.

Meta-models are a powerful tool for the solution of complex realistic problems in stochastic dynamics. L. Pichler et al. present a mode-based meta-model for the reliability assessment of linear dynamical systems (Chapter 8). The reliability analysis is carried out through the application of the meta-model for estimating the modal properties which are then subsequently required to perform a mode-superposition analysis. The proposed approach is proved to be accurate and computationally efficient in both cases of deterministic and stochastic loading. The verification of the results for the reliability assessment carried out with the meta-model is performed with a full finite element (FE) analysis and shows a good agreement of the response over the whole duration of the stochastic ground acceleration.

The effect of uncertain system properties on structural response and reliability is examined by G. Stefanou and M. Fragiadakis in Chapter 9. An efficient approach combining MCS with translation process theory is presented for assessing the non-linear stochastic response and reliability of a steel moment-resisting frame subjected to transient seismic actions. The structure is modeled with a mixed fiber-based beam-column element, whose kinematics is based on the natural mode method. The adopted formulation leads to the reduction of the computational cost required for the calculation of the element stiffness matrix, while increased accuracy compared to traditional displacement-based elements is achieved. The uncertain parameters of the problem are the Young modulus and the yield stress, both described by homogeneous non-Gaussian translation stochastic fields. Under the assumption of a pre-specified power spectral density function of the stochastic fields, a parametric

investigation is carried out providing useful conclusions regarding the influence of the correlation length of the stochastic fields on the response variability and reliability of the frame.

Although reliability analysis methods have matured in recent years, the number of available techniques to evaluate the risk of complex structural systems is limited. In Chapter 10, F. Petrin et al. investigate the problem of Aeolian risk assessment of slender structures in the framework of performance-based wind engineering. A classification of the main sources of uncertainty is attempted and the importance of some of them (epistemic uncertainty of the aerodynamic coefficients, model uncertainty of aeroelastic forces, influence of model uncertainty on risk assessment) is examined with reference to an example case, a long span suspension bridge.

The concept of evolutionary power spectrum is important in the context of stochastic simulation of earthquake ground motions. Chapter 11 by D. Schillinger and V. Papadopoulos presents a novel approach called “method of separation” for the accurate estimation of evolutionary power spectra of non-homogeneous strongly narrow-band stochastic fields encountered in various engineering applications. The method is developed for power spectra separable in space-frequency but can also efficiently treat a class of non-separable spectra. The method of separation is based on the geometrical similarity of functions. It leads to the estimation of evolutionary power spectra from a series of samples based only on sample mean square and on an estimate of the homogeneous Fourier power spectrum. The limitations of existing methods (short-time Fourier, wavelet and Wigner-Ville transforms) are avoided and a very good localization in both space and frequency is achieved. Numerical examples including analytical benchmark spectra as well as spectra corresponding to measured geometric imperfections of I-section beams illustrate the superiority of the proposed approach in comparison to other established techniques.

An energy-based envelope function is proposed in Chapter 12 by S. Sgobba et al. enabling to characterize the time-varying amplitude of strong ground motion and to generate realistic stochastic accelerograms corresponding to a given earthquake scenario. This envelope function is directly related to the Arias intensity of the ground motion and has a functional form similar to that of a lognormal probability density function. In conjunction with suitable peak factors, the envelope function may also be used to predict the distribution of PGA values corresponding to a given earthquake scenario.

The development of robust and efficient solution algorithms is crucial for the treatment of large-scale realistic problems involving uncertainties. A novel numerical algorithm for the solution of stochastic partial differential equations (SPDEs) arising in the context of stochastic mechanics is presented by H. Matthies and E. Zander in Chapter 13. The algorithm exploits the natural tensor product structure between basis vectors describing the physical/deterministic behavior and a basis describing the stochastic response of the system. A sparse representation of input and output is achieved using singular value decomposition (SVD). This sparse format is preserved throughout the computation and is also used to reduce the amount of computation. A numerical example involving the 1D diffusion equation is presented

to illustrate the efficiency of the approach. It is shown that for a sufficiently accurate representation of the solution, the required storage is reduced by 90% and by 50–80% as intermediate storage is concerned.

The problem of dynamic stability is treated in Chapter 14. J. Náprstek and C. Fischer investigate the dynamic stability behavior of a non-linear auto-parametric system with three degrees of freedom (used as a simple mathematical model of a slender structure on elastic subsoil exposed to strong vertical excitation). Certain intervals of the excitation frequency are identified where the semi-trivial solution loses its dynamic stability and the strong horizontal response components become important. Post-critical states of both deterministic and chaotic type are thoroughly analyzed with respect to excitation frequency and amplitude. It is observed that the response amplitudes can remain in acceptable limits when adequate excitation (in terms of frequency and amplitude) acts throughout a short time interval. It is mostly not the case when the excitation has an infinite duration and the response is approaching to the steady state.

The last two chapters of the book are devoted to the rapidly evolving area of stochastic modeling of heterogeneous materials. A. Ibrahimbegovic et al. (Chapter 15) discuss the failure analysis of civil engineering structures built of heterogeneous materials. The structured FE mesh approach (consisting of constant stress triangular element that can contain two different phases and phase interface) is proved to be much more efficient than the non-structured mesh representation of material heterogeneities for the case of 2D two phase materials. This feature is of special interest for probabilistic analysis, where a large amount of computation is needed in order to provide the corresponding statistics. One such case of probabilistic failure analysis is also considered in this chapter, where the geometry of the phase interface remains uncertain since it is obtained as the result of the Gibbs random process. This computation is further used to provide the appropriate probabilistic description of material parameters of phenomenological model of localized failure in terms of correlated random fields. Subsequent Monte Carlo computations of failure phenomena in simple tension test performed with such probabilistic phenomenological model clearly show the capability of the presented approach to recover the size effects anywhere within a range between the two classical bounds which are Continuum Damage Mechanics and Linear Fracture Mechanics.

The book closes with a stochastic mechanics approach by D. Schwarzer and C. Proppe for the calculation of the statistics of eigenfrequencies in heterogeneous beams made of metal foam (Chapter 16). The spatial variation of the linear elastic material properties of metal foam is quantified using random fields generated by the spectral representation and Karhunen-Loève expansion methods. The probability distribution and the correlation structure of the random fields are derived from data. MCS of eigenvibration problems of beams of different length are performed. The computed eigenfrequencies are compared to experimental data. From this comparison, it can be concluded that the heterogeneity of the material microstructure affects the macroscopic properties. The mean values of the eigenfrequencies are smaller than those of beams with homogeneous material of similar mean material

properties. A scatter of the eigenfrequencies is also observed, a fact that has to be taken into account when using parts consisting of metal foam.

The book editors would like to express their deep gratitude to all contributors for their active participation in the COMPDYN 2009 and SEECCM 2009 Conferences and for the time and effort devoted to the completion of their contributions to this volume. Special thanks are also due to the reviewers for their constructive comments and suggestions which enhanced the quality of the book. Finally, the editors would like to thank the personnel of Springer for their most valuable support during the publication process. It is hoped that this book will convey the reader to the excitement, advances and challenges the computational stochastic dynamics community faces in the near future.

June 2010

Manolis Papadrakakis
George Stefanou
Vissarion Papadopoulos

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Model Reduction and Uncertainties in Structural Dynamics

Gerhart I. Schuëller

in Cooperation with Barbara Goller and Helmut J. Pradlwarter

Abstract Model reduction procedures for the purpose of reducing computational efforts in structural analysis are already well developed and widely used to compute the dynamic response of complex structural systems. For example the Guyan reduction might essentially reduce the size of the structural matrices, while component mode synthesis provides a means to consider first simpler substructures and to use its modal properties for deriving a reduced global structural model for computing the dynamic response. Moreover, component mode synthesis allows for an assembly of large finite element models consisting of substructures established by different working groups and hence has significant advantages in the design cycle.

The above mentioned, frequently used finite element (FE) reduction procedures assume perfect, i.e. deterministic knowledge of all structural properties. However, the assumption of deterministically known structural properties is in most practical, real world cases, not realistic. Many items (e.g. stiffness, mass and damping parameters, etc.) incorporated in the mathematical FE model of a structure are uncertain, i.e. are either not precisely known or might reveal unpredictable random behavior.

This paper will give an overview of the well established deterministic reduction techniques and approaches for the efficient uncertainty propagation. Finally, the recent advances in the combination of these two fields are reviewed and methodologies for the consideration of uncertainty when using model reduction schemes are presented. The model reduction schemes considering uncertainties, which usually involve some simplifying assumptions, similar to their deterministic counterparts, are compared with the reference solution as obtained by direct Monte Carlo simulation (MCS), where the deterministic FE reduction scheme is applied together with randomly generated structural properties.

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1 Introduction

Model reduction is a commonly used technique in structural dynamics to facilitate the analysis of complex structural systems. The primary purpose of model reduction techniques are the establishment of models which are simpler to manage and faster to analyze. Especially in a first stage of the design, approximate responses are often sufficient for further design decisions. Nowadays, and surely in the near future, structural analysis is almost exclusively performed by applying Finite Element software. Automated mesh generation usually produce a detailed fine mesh which involves a large number of degrees of freedom to represent local stress concentration with sufficient fidelity. In structural dynamics, however, the response of many degrees of freedom can be safely represented by so called master degrees of freedom without noteworthy loss in accuracy. For large complex structures, the modeling of substructures and its global assembly provides many advantages, computationally and organizationally. For this purpose, various component mode synthesis procedures have been developed. These numerical methods speed up the development cycle and are especially useful in cases where many modifications of a structure need to be analyzed.

The above mentioned methods are also very useful in case the robustness of the response prediction should be investigated. This issue of robustness arises from uncertainties regarding the adequacy of the mathematical (FE) model and from the parameter uncertainties applied within the FE model. Unfortunately, the adequacy of a FE-model for its specified purpose is generally difficult to validate and often requires expensive measurements for an objective decision process. The study of the robustness of the response predictions by incorporating uncertainties of the input parameters, however, is a feasible task. Structural engineers are expected to know which parameters are uncertain and to have some information on the variability of these uncertain quantities, e.g. lower and upper bounds, mean value, standard deviation and similar. Translating the available information into probability distributions allows to study the resulting uncertainties of the response.

The model reduction techniques developed in the context of deterministic structural analysis provide useful techniques when evaluating the resulting uncertainties of the structural response. Since uncertainty propagation is generally a computationally intensive task, these techniques can be employed to keep the efforts computationally feasible and also within reasonable time constraints. These techniques need to be adjusted to the special needs in stochastic structural mechanics where the focus will be on stochastic structural dynamics. To study the response of uncertain linear dynamical structural systems, most attention has been paid to the so called random eigenvalue problem. Although, its efficient solution does not cover non-linear structural problems, it is certainly of central interest for many practical applications where larger linear structural FE model are used. There exist a large variety of approaches of which representative methods are selected to provide an informative overview on these methods.

2 Deterministic Procedures to Reduce Models

2.1 General Remarks

The increasing need of many industrial fields for highly accurate predictions gives rise to the development of finite element models of high complexity. The size of these models may lead to enormous computational efforts which may be even beyond the limits of hardware resources. Hence, it is necessary to reduce the computational costs by establishing surrogate models which approximate the full model and truncate unnecessary information. Another approach consists of the division of the model into substructures. The most commonly used approaches for model reduction will be discussed in the following subsections.

2.2 Guyan Reduction

Guyan reduction [21] is a substructuring approach which reduces the size of the model in a smaller problem by relating the displacements of the so-called slave degrees of freedom (DOFs) to the master DOFs. Hence, the governing equation in static analysis, $\mathbf{K}\mathbf{u} = \mathbf{F}$, can be written as

$$\begin{bmatrix} \mathbf{K}_{SS} & \mathbf{K}_{SM} \\ \mathbf{K}_{MS} & \mathbf{K}_{MM} \end{bmatrix} \begin{bmatrix} \mathbf{u}_S \\ \mathbf{u}_M \end{bmatrix} = \begin{bmatrix} \mathbf{F}_S \\ \mathbf{F}_M \end{bmatrix}, \quad (1)$$

where the indices S and M refer to the slave and master DOFs, respectively. Under the condition that the slave DOFs are unloaded, the displacements of the master DOFs are given by

$$\mathbf{u}_M = [\mathbf{K}_{MM} - \mathbf{K}_{MS}\mathbf{K}_{SS}^{-1}\mathbf{K}_{SM}]^{-1} \mathbf{F}_M. \quad (2)$$

The slave DOFs are related to the master DOFs through

$$\mathbf{u}_S = -\mathbf{K}_{SS}^{-1}\mathbf{K}_{SM}\mathbf{u}_M. \quad (3)$$

Hence, the static analysis can be partitioned into two smaller problems, where the solutions for $\mathbf{u}_S$ and $\mathbf{u}_M$ involve no approximations.

In dynamics, the application of the Guyan reduction to the determination of the structural response $\mathbf{u} = \mathbf{x}e^{i\omega t}$ of an undamped system with harmonic excitation $\mathbf{F}e^{i\omega t}$ of the form

$$\begin{bmatrix} \mathbf{K}_{SS} - \omega^2\mathbf{M}_{SS} & \mathbf{K}_{SM} - \omega^2\mathbf{M}_{SM} \\ \mathbf{K}_{MS} - \omega^2\mathbf{M}_{MS} & \mathbf{K}_{MM} - \omega^2\mathbf{M}_{MM} \end{bmatrix} \begin{bmatrix} \mathbf{x}_S \\ \mathbf{x}_M \end{bmatrix} = \begin{bmatrix} \mathbf{F}_S \\ \mathbf{F}_M \end{bmatrix} \quad (4)$$

yields the following solution for the slave DOFs

$$\mathbf{x}_S = -[\mathbf{K}_{SS} - \omega^2 \mathbf{M}_{SS}]^{-1} [\mathbf{K}_{SM} - \omega^2 \mathbf{M}_{SM}] \mathbf{x}_M. \quad (5)$$

This equation enforces a matrix inversion at each frequency step ω which leads to a high computational effort. Hence, an approximation of Eq. 5 is performed such that the contribution of the mass matrix is neglected, which leads to

$$\mathbf{x}_S = -\mathbf{K}_{SS}^{-1} \mathbf{K}_{SM} \mathbf{x}_M. \quad (6)$$

The application of Guyan reduction to dynamical systems therefore introduces some approximation errors. Approaches for the improvement of the Guyan condensation method in dynamics can be found in e.g. [5].

2.3 Component Mode Synthesis

The basic idea of component mode synthesis (CMS) is the division of the full model into a set of substructures and to describe these component models in terms of some basis vectors. By enforcing equilibrium and compatibility along component interfaces, the substructures are assembled to the global model, which exhibits a considerably smaller size than the original problem. The most common approaches in CMS, which was originally introduced by Hurty in 1965 [23], are the fixed interface method with constraint modes (Craig-Bampton method) [9–11] and the free interface method with attachment modes [12, 13].

2.3.1 Fixed Interface Craig-Bampton Method

The structural matrices are partitioned according to the boundary degrees of freedom (index B) and the interior degrees of freedom (index I) which leads to the following form of the undamped equation of motion of one component α

$$\begin{bmatrix} \mathbf{M}_{II} & \mathbf{M}_{IB} \\ \mathbf{M}_{BI} & \mathbf{M}_{BB} \end{bmatrix}^{(\alpha)} \begin{bmatrix} \ddot{\mathbf{u}}_I \\ \ddot{\mathbf{u}}_B \end{bmatrix}^{(\alpha)} + \begin{bmatrix} \mathbf{K}_{II} & \mathbf{K}_{IB} \\ \mathbf{K}_{BI} & \mathbf{K}_{BB} \end{bmatrix}^{(\alpha)} \begin{bmatrix} \mathbf{u}_I \\ \mathbf{u}_B \end{bmatrix}^{(\alpha)} = \begin{bmatrix} \mathbf{F}_I \\ \mathbf{F}_B \end{bmatrix}^{(\alpha)}. \quad (7)$$

The physical coordinates $\mathbf{u}^{(\alpha)}$ can be represented by component generalized coordinates, denoted $\mathbf{p}^{(\alpha)}$ using the transformation matrix $\mathbf{B}^{(\alpha)}$

$$\mathbf{u}^{(\alpha)} = \mathbf{B}^{(\alpha)} \mathbf{p}^{(\alpha)}. \quad (8)$$

In the Craig-Bampton method, the matrix $\mathbf{B}^{(\alpha)}$ is composed of fixed interface modes and constraint modes, where the fixed interface modes are obtained solving the eigenvalue problem

$$\mathbf{K}_{II} \boldsymbol{\Phi} = \mathbf{M}_{II} \boldsymbol{\Lambda} \boldsymbol{\Phi}. \quad (9)$$

The constraint modes are defined as the static reactions of the interior DOFs due to a unit displacement of one boundary DOF and zero displacement of the remaining boundary DOFs, and are given by

$$\boldsymbol{\psi} = \begin{bmatrix} -\mathbf{K}_{II}^{-1} \mathbf{K}_{IB} \\ \mathbf{I}_{BB} \end{bmatrix}. \quad (10)$$

Hence, the transformation matrix $\mathbf{B}^{(\alpha)}$ in the Craig-Bampton method is given by

$$\mathbf{B}^{(\alpha)} = \begin{bmatrix} \boldsymbol{\Phi}_k & -\mathbf{K}_{II}^{-1} \mathbf{K}_{IB} \\ \mathbf{0} & \mathbf{I}_{BB} \end{bmatrix}, \quad (11)$$

where the index k refers to the number of kept normal modes.

The transformation of the mass and stiffness matrix into the component modal space yields the component modal mass and stiffness matrix

$$\boldsymbol{\mu}^{(\alpha)} = \mathbf{B}^{(\alpha)T} \mathbf{M}^{(\alpha)} \mathbf{B}^{(\alpha)} = \begin{bmatrix} \mathbf{I}_{kk} & \mathbf{m}_{IB} \\ \mathbf{m}_{BI} & \mathbf{m}_{BB} \end{bmatrix}, \quad \boldsymbol{\kappa}^{(\alpha)} = \mathbf{B}^{(\alpha)T} \mathbf{K}^{(\alpha)} \mathbf{B}^{(\alpha)} = \begin{bmatrix} \boldsymbol{\Lambda}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{k}_{BB} \end{bmatrix} \quad (12)$$

The matrices $\mathbf{m}_{BB}$ and $\mathbf{k}_{BB}$ are the constraint modal mass and stiffness matrices and the matrix $\mathbf{m}_{IB}$ is a coupling matrix. The matrix $\boldsymbol{\Lambda}_k$ is a diagonal matrix composed of the eigenvalues that are retained. In case of many boundary DOFs as it appears in applications with line and surface coupling, the size of the FE-model can be dominated by the boundary DOFs and it might be desirable to reduce the size of the constraint and coupling matrices. One approach is given by the use of so-called characteristic constraint modes which form a highly reduced basis for the representation of the constraint modes [7].

As a next step, the substructure matrices $\alpha = 1, \dots, N_c$ are assembled to the global mass and stiffness matrix in the modal space.

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}^{(1)} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\mu}^{(2)} & & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \boldsymbol{\mu}^{(N_c)} \end{bmatrix}, \quad \boldsymbol{\kappa} = \begin{bmatrix} \boldsymbol{\kappa}^{(1)} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\kappa}^{(2)} & & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \boldsymbol{\kappa}^{(N_c)} \end{bmatrix} \quad (13)$$

The displacements at the juncture DOFs of two components α and β which have a common boundary are constrained by

$$\mathbf{u}_B^{(\alpha)} = \mathbf{u}_B^{(\beta)}. \quad (14)$$

Hence, a relation to impose the coupling constraint has to be established such that

$$\mathbf{p} = \mathbf{S}\mathbf{q}, \quad (15)$$

where the vector $\mathbf{p}$ contains the complete set of generalized components using all components of $\mathbf{p}^{(\alpha)}$, $\alpha = 1, \dots, N_c$ and the vector $\mathbf{q}$ contains the set of independent generalized coordinates (i.e. the redundant coordinates are excluded). The matrix $\mathbf{S}$ is then used to establish the global mass and stiffness matrices

$$\mathbf{M}_{\text{red}}^{\text{glob}} = \mathbf{S}^T \boldsymbol{\mu} \mathbf{S}, \quad \mathbf{K}_{\text{red}}^{\text{glob}} = \mathbf{S}^T \boldsymbol{\kappa} \mathbf{S}. \quad (16)$$

The reduced eigenvalue problem can now be solved

$$\mathbf{K}_{\text{red}}^{\text{glob}} \boldsymbol{\Phi}_{\text{red}} = \mathbf{M}_{\text{red}}^{\text{glob}} \boldsymbol{\Lambda}_{\text{red}} \boldsymbol{\Phi}_{\text{red}}. \quad (17)$$

2.3.2 Component Modes for Unconstrained Components

An alternative approach to define the transformation matrix $\mathbf{B}^{(\alpha)}$ is given by the use of the free normal modes in combination with the attachment modes. The free interface modes are obtained by solving the eigenvalue problem which involves the full component matrices (i.e. corresponding to interior and boundary DOFs). The attachment modes are defined as the static deformation of the interior DOFs due to a unit force applied at one boundary DOF and with the remaining DOFs unloaded. In case the component exhibits rigid body motion, it must be sufficiently constrained and the rigid body modes are then added separately. An alternative approach of defining attachment modes in combination with rigid body motion consists of the use of inertia relief modes, where the details can be found in [31, 43].

Although the discussion of the inclusion of damping is beyond the scope of this presentation it should be noted that also the damping matrix can be included in the CMS approach (see [4, 22]), where improvements of these original versions can be found, e.g. in [14]. In case of general viscous damping, complex component mode synthesis is required [37, 42], whereas applications in context of non-linear hysteretic elements are shown in [40].

2.4 Meta-Models

Apart from substructuring techniques the use of surrogate models offers the possibility to reduce the computational efforts considerably while still achieving highly accurate approximations. Response surfaces were introduced by Box and Wilson in 1951 [6] and developed into an indispensable tool for many scientific and technical applications in the development, improvement and optimization of the design. The scope of such a surrogate model is the representation of a relationship between several input variables and an output quantity. There is a wealth of different approaches available where the most commonly used meta-models are based on linear or polynomial regression, on a least-squares formulation or on kriging and radial basis functions. Comparisons of different approaches and examples for applications

can be found in e.g. [15, 20, 24, 45, 54]. A thorough treatment of various approaches for the formulation of response surfaces can be found, e.g. in [8, 34].

Similarly to the response surface method, a so-called *meta-model*, which represents a functional relation between the input parameters and the model properties, has been developed in order to approximate the eigenfrequencies and eigenvectors of large dynamical systems (see e.g. [39]). This meta-model can be used for the assessment of the variability of the structural response and also for optimization and reliability analysis.

The basic idea of this approach is the approximation of an eigenvector $\phi_k^{(j)}$ in the j -th simulation as a linear combination of the neighboring modes of a reference solution

$$\phi_k^{(j)} = \sum_{i=k-m_k}^{k+m_k} \alpha_{ki}^{(j)} \phi_i^{(0)}, \quad (18)$$

where the parameters $\alpha_k = \{\alpha_{k,i}\}_{i=k-m_k}^{k+m_k}$ represent the coefficients of the linear combinations of the reference modes $\Phi^{(0)}$ and are obtained by projecting the vector $\phi_k^{(j)}$ onto the subspace spanned by the reference eigenvectors. The assumption is that in case of small variability of the input parameters in combination with well separated modes the eigenvectors differ only slightly from the nominal solution which was also shown in e.g. [53]. The corresponding eigenfrequency is given by

$$\omega_k^{(j)} = (1 + \Delta) \omega_k^{(0)}. \quad (19)$$

Hence, the modal characteristics of each simulation can be described by $\alpha_{ki}^{(j)}$, Δ and the reference solutions $\omega_k^{(0)}$, $\Phi^{(0)}$. With respect to the relationship between these quantities and the structural input parameters, a linear approximation suffices to be sufficiently accurate. The matrix representing the functional relation is based on a calibration set and can be calculated using several procedures, such as the least-squares method, regression, optimization or soft computing methods.

3 Stochastic Analysis

3.1 Uncertainty Modeling

Engineers are aware that their established models represent only an approximation to reality and that there are various sources of uncertainties in the modeling process, such as the uncertainties in the model parameters and the simplifying assumptions made to reduce the complexity of the model. Hence, in order to give confident predictions of the target structural responses the consideration of uncertainties becomes an indispensable step in the design process. It has been recognized already in the

1920s that the modeling process should account for a quantitative assessment of the uncertainties [32]. Comprehensive surveys of the developments of theoretical aspects and their practical applicability can be found, e.g. in [46–49].

Most uncertainty quantification schemes are based on the description of the uncertainties as random variables, stochastic processes or random fields. A random variable X is described by its probability density function $f(x)$ (PDF) and the probability that the parameter takes a value within a certain interval $[a, b]$ is given by the integral

$$P(a < X < b) = \int_a^b f(x) dx, \quad (20)$$

where this formula can be extended to multi-dimensional space. Typical examples for random variables are the Young's modulus, the thickness, etc.

A random variable $f(x, t)$ which varies in time is called a stochastic process. Examples are given by the uncertainties associated with wind, earthquakes, etc. A dependence of the uncertain quantity on the spatial coordinate is modeled by a random field, e.g. a spatially varying Young's modulus.

The variation of a random variable in time or space can be represented by a Karhunen-Loève (KL) expansion which is given for a Gaussian random variable in the form

$$X(\varphi) = x^{(0)}(\varphi) + \sum_{j=1}^m \xi_j x^{(j)}(\varphi), \quad (21)$$

where the variable φ may refer to the time or a spatial coordinate. The functions $x^{(0)}(\varphi)$ and $x^{(j)}(\varphi)$ are deterministic functions and ξ_j is a realization of a standard normal distribution. The definition of the basis functions $x^{(j)}(\varphi)$ is performed such that only a few terms m are necessary to represent $X(\varphi)$ with high accuracy. In structural dynamics, the Karhunen-Loève expansion is mainly used for the representation of random fields and random processes, applications in the characterization of the structural response can be found in e.g. [55], where a reduction in phase space is performed by using KL-expansion.

In addition to approaches which consider uncertainties on the physical level, i.e. by considering the scatter of the physical model parameters, an approach has recently been proposed which takes uncertainties on a global level into account. This method is referred to as a non-parametric approach [51, 52] and defines the matrix entries of the reduced numerical model as random variables. This reduced numerical model is obtained with the truncated modal basis of the full nominal model.

3.2 Simulation Techniques

Several approaches have been developed in order to capture these different types of uncertainties. The most versatile approach to capture uncertainties is the Monte

Carlo simulation (see e.g. [16]). However, due to the associated large computational efforts, already the analysis of moderate sized problems may meet the limits of computational feasibility. Hence, large research efforts have been ongoing towards the minimization of the analysis time.

One way to accomplish this goal is the development of advanced simulation algorithms which are directed towards “controlled” sampling, i.e. towards guiding the samples into the domain of interest, which is e.g. the failure domain in case of reliability analysis. In case the domain of interest is known or can be computed by gradient based algorithms, methods such as importance sampling (see e.g. [50]) or line sampling [29] are very efficient procedures. In other cases, as e.g. for strongly non-linear responses, subset simulation [2] is an accurate and powerful technique.

The second approach in order to reduce the computational costs is the parallelization of the analysis. The importance of parallel computing in stochastic structural mechanics has long been emphasized (see e.g. [17]). In the field of structural dynamics, Johnson et al. [25] investigated the performance of massively parallel platforms for the Monte-Carlo simulation of stochastic dynamics problems. Applications of parallel processing in context with uncertainty quantification in structural dynamics can be found in e.g. [26, 54] and referring to reliability analysis in e.g. [38].

The third approach is concerned with the minimization of the computational costs of one simulation. This can be achieved by using local approximation methods based on the perturbation method or Neumann expansion series which are widely used in computational stochastic analysis [28]. The perturbation method is based on a Taylor expansion of the structural matrices, the load vector and the vector of the structural response. The computational efforts grow with the number of uncertain parameters and with the order of the expansion. In case of small coefficients of variation, a linear approximation of the response turns out to predict the structural response sufficiently accurately. The Neumann expansion takes advantage – similarly to the perturbation approach – from a reference solution and is based on the approximation of the solution using a few basis vectors which span the preconditioned stochastic Krylov subspace as shown in [35]. Hence, the Neumann method constitutes a special case of a stochastic reduced basis method as proved in [27].

Another stochastic reduced basis method which approximates the solution in the random dimension – and not in space as described above – is given by the Polynomial Chaos decomposition scheme [18]. The key ingredient of the chaos expansion is to represent the random entities by an orthogonal basis defined in a Hilbert space. Applications of Polynomial Chaos in dynamics, specifically for the random eigenvalue problem, are shown in e.g. [19, 56]. A comparison of different subspace projection techniques can be found in [44].

The goal of reducing the computational costs of one single simulation can also be reached by using the substructure techniques described in Sect. 2. Hence, recent advances and new developments in the combination of uncertainty quantification methods and model reduction techniques are addressed in the following.

4 Model Reduction Taking into Account Uncertainties

4.1 Introduction

Component mode synthesis represents an appealing framework for the quantification of uncertainty in build-up structures due to the computational savings that occur in each simulation [3]. In addition, the substructures are usually statistically independent since they are manufactured by different companies and finally assembled. Hence, this makes it possible to independently analyze the components with the option of applying different uncertainty propagation procedures to the components.

Component mode synthesis is also particularly advantageous for the application of perturbation approaches in order to estimate the variability at the global level. Due to the almost linear relationship between the structural parameters and the component modal properties, as well as between the component modal properties and global modal properties, it is possible to accurately approximate the variability of the global modal properties and frequency response functions by linear approximations. This local modal/perturbational approach was proposed in [30]. Another approach for the assessment of the variability of the eigenfrequency and eigenvectors using the advantages of substructuring is presented in [41], where a special form of CMS is proposed which is oriented towards an efficient computation of eigenvectors and eigenvalues.

A further possibility for the consideration of uncertainty that is offered by CMS methods is based on the Karhunen Loève expansion of the substructure matrices where the sparse structure of the modal substructure matrices leads to a computationally efficient uncertainty propagation method. This novel approach will be discussed in detail in the next section.

In addition, a second approach will be proposed that uses the advantages of substructuring by randomly combining independently simulated substructures. Hence, in addition to the probability density functions of the physical parameters, also the random combination, expressed by discrete uniform distributions for the set of substructure matrices, introduces variability in order to capture the uncertainty by which the structure is affected. Finally, the combination of the approximation methods and substructuring constitutes a straight forward combination of advanced techniques for uncertainty propagation and model reduction schemes. These approaches will be described in the following.

4.2 Karhunen-Loève Expansion of the Substructure Matrices

4.2.1 Introduction

The idea of this approach consists of capturing uncertainties not on the *physical level* but rather in the *modal space* of each substructure. The special form of the Craig-Bampton substructure matrices with its high sparsity (see Eq. 12) makes it

computationally advantageous to define the entries of the substructure matrices in the modal space as random variables and to reproduce the variability of the physical parameters on the matrix level.

In order to calibrate the uncertain reduced substructures, samples of substructures are generated by considering the scatter of the physical parameters. Based on this sample set of each substructure, the second order statistics of the entries of the reduced matrices can be estimated. The vectors of the mean values and the covariance matrices are then used to represent the entries using a Karhunen-Loève expansion as introduced in Sect. 3 for random processes and random fields. As opposed to random processes and random fields the Karhunen-Loève representation does not discretize the process in time or space but in the random dimension by transforming a number of correlated matrix entries into a smaller number of uncorrelated variables (i.e. principal components).

4.2.2 Calibration Set for the Structural Matrices

As a first step, a set of substructure matrices $\mu_j^{(\alpha)}$ and $\kappa_j^{(\alpha)}$ with the corresponding transformation matrix $\mathbf{B}_j^{(\alpha)}$, for $j = 1, \dots, J$ is generated, where J denotes the number of simulations forming the calibration set. This yields the vectors $\hat{\mathbf{x}}_j^\mu$ and $\hat{\mathbf{x}}_j^\kappa$ which are composed of the non-zero entries of the modal mass and stiffness matrix of one simulation, where the symmetry of the matrices leads to a reduced number of entries to be stored.

When generating the calibration set for the modal mass matrix it should be noted that in contrast to the matrices $\mathbf{m}_{BB}$ and $\mathbf{k}_{BB}$ the coupling matrix $\mathbf{m}_{IB}$ is a function of the normal modes. Hence, the entries of this matrix obtained from two different simulations are only physically comparable if the sequence of the mode shapes is the same in both simulations. Hence, mode pairing has to be adopted, which can be performed using e.g. the widely applied modal assurance criterion (MAC) that has been introduced in [1].

4.2.3 Calibration Set for the Matrix of Eigenvectors and Constraint Modes

The number of non-zero entries of the transformation matrix $\mathbf{B}^{(\alpha)}$ is too high in order to efficiently use the vectors $\hat{\mathbf{x}}_j^\phi$ and $\hat{\mathbf{x}}_j^\psi$ for the calibration set since the computation of the eigensolution of the covariance matrix may lead to an infeasible computational effort for larger systems. Hence, the idea is to represent the random eigenvectors ϕ_j and constraint modes ψ_j by linear combinations of basis vectors, i.e.

$$\phi_j = \Phi^{\text{ref}} \hat{\mathbf{Q}}_j, \quad \psi_j = \Psi^{\text{ref}} \hat{\mathbf{P}}_j \quad (22)$$

These basis vectors Φ_j and Ψ_j are an approximation to the full space of eigenvectors and constraint modes, which are obtained in every simulation. The coefficients

$\hat{\mathbf{Q}}_j$ and $\hat{\mathbf{P}}_j$ are obtained by solving a least squares problem. The construction of the reference subspaces $\hat{\Phi}^{\text{ref}}$ and $\hat{\Psi}^{\text{ref}}$ is based on the extraction of important components of the set of random eigenvector bases and random constraint mode bases. The extraction of the important component can be carried out by projecting the eigenvectors and constraint modes of the current simulation onto the subspace $\mathbf{X}$, i.e.

$$\Pi_{\mathbf{X}}(\phi_j) = \sum_{i=1}^k \frac{\langle \phi_j, \mathbf{x}_i \rangle}{\|\mathbf{x}_i\|^2} \mathbf{x}_i, \quad (23)$$

The initialization of the subspace can be performed using the orthonormal eigenvectors of the first simulation. The same procedure is employed for the constraint modes. If the eigenvector or constraint mode can not be represented with a certain accuracy by the current basis vectors, then the subspace is augmented.

4.2.4 Karhunen-Loève Representation of the Matrices

The vectors forming one simulation $\hat{\mathbf{x}}_j^\mu$, $\hat{\mathbf{x}}_j^\kappa$, $\hat{\mathbf{x}}_j^Q$ and $\hat{\mathbf{x}}_j^P$ of each substructure (the index α has been omitted) are assembled to one simulation vector $\hat{\mathbf{x}}_j^{(\mu,\kappa,Q,P)}$. The vectors $\hat{\mathbf{x}}_j^{(\mu,\kappa,Q,P)}$, $j = 1, \dots, J$ form the calibration set of each substructure and are used for the estimation of the mean vector $\bar{\mathbf{x}}^{(\mu,\kappa,Q,P)}$ and the covariance matrix $\mathbf{C}^{(\mu,\kappa,Q,P)}$. An eigenvalue solution of the covariance matrix yields the basis vectors for reproducing the uncertainty in the calibration set. The most important directions are associated with the largest eigenvalues which leads to the following representation of the matrix entries for any $k > J$:

$$\mathbf{x}_k^{(\mu,\kappa,Q,P)} = \bar{\mathbf{x}}^{(\mu,\kappa,Q,P)} + \sum_{i=1}^{N_{KL}} \xi_i \sqrt{\lambda_i} \phi_i, \quad (24)$$

where λ_i and ϕ_i denote the i th eigenvalue and eigenvector of the covariance matrix with $N_{KL} \ll N$, and N is the length of the vector $\mathbf{x}_j^{(\mu,\kappa,Q,P)}$.

4.2.5 Assembly of the Global Matrices

The representation of the substructure matrix entries by a KL-expansion allows for an efficient simulation of the matrix entries. Hence, in each simulation, the vector $\mathbf{x}^{(\mu,\kappa,Q,P)}$ is simulated as described above. Then, the modal mass matrix μ and the modal stiffness matrix κ can be established according to the structure in Eq. 12. These substructure matrices are assembled to the global reduced matrices according to the Craig-Bampton reduction scheme (see Eqs. 13–16).

An example-based verification of this approach where the results are compared with direct Monte Carlo simulation will be presented in Sect. 5.

4.3 Random Combination of Substructures

In the approach presented here, a random sampling strategy is applied in order to obtain a statistical assessment of the quantity of interest using only a moderate sample size of e.g. 30–50 samples. The idea consists of the generation of a pool of random substructure matrices and to combine these substructure matrices randomly.

More specifically, $J^{(\alpha)}$ structural matrices of the substructure α are generated and an equal probability is assigned to each realization j

$$p_j^{(\alpha)} = \frac{1}{J^{(\alpha)}}. \quad (25)$$

The global structural matrices are composed of the substructures $\alpha = 1, \dots, N_C$, where N_C denotes the number of components. These samples are generated using direct Monte Carlo simulation. In order to represent the variability with a few samples it can also be advantageous to use Latin Hypercube sampling. The random combination of the substructures introduces some additional variability in the structural response which makes the approach particularly advantageous in case of many substructures. This method aims to imitate the randomness associated with random sampling. Once the set of substructures is selected, the global reduced structural matrices are assembled by applying the Craig-Bampton method (see Eqs. 13–16). It should be noted that this approach can also be used if mode switching occurs and is thus also applicable for the higher frequency range. An example-based verification of this approach will be given in Sect. 5.

4.3.1 Stochastic Reduced Basis Approximation (SRBA)

The simulation of the set of substructures needed for the above approach may lead to large computational efforts if dealing with complex substructures. The most time-consuming part concerns the solution of the eigenvalue problem. Therefore it is advantageous to apply approximate solution schemes. Among these approximation methods, the Stochastic Reduced Basis Approximation (SRBA) introduced in [36], which constitutes a reduced basis formulation for the efficient solution of large-scale eigenvalue problems, will be in the focus of this manuscript. The eigenvalue problem is given by

$$\left[\mathbf{K}_0 + \sum_{i=1}^p \theta_i \mathbf{K}_i \right] \boldsymbol{\phi}(\boldsymbol{\theta}) = \lambda(\boldsymbol{\theta}) \left[\mathbf{M}_0 + \sum_{i=1}^p \theta_i \mathbf{M}_i \right] \boldsymbol{\phi}(\boldsymbol{\theta}), \quad (26)$$

where $\mathbf{K}_0$ and $\mathbf{M}_0$ denote the nominal mass and stiffness matrices, $\sum_{i=1}^p \theta_i \mathbf{K}_i$ and $\sum_{i=1}^p \theta_i \mathbf{M}_i$ are the perturbations in the mass and stiffness matrix and $\boldsymbol{\phi}(\boldsymbol{\theta})$ and $\lambda(\boldsymbol{\theta})$ are the modal properties of the perturbed system. In the present application, these structural matrices do not refer to the full system, but to the substructures of the investigated structure.

The basic idea of this approach consists of the representation of the basis vectors for approximating the random eigenvalue solution by the eigenvectors of the nominal solution and their first order derivatives. Therefore, the approximation for a perturbed eigenvector ϕ can be written as

$$\phi(\theta) = \zeta_1(\theta)\phi_0 + \zeta_2(\theta) \sum_{i=1}^p \theta_i \frac{\partial \phi}{\partial \theta_i}, \quad (27)$$

where θ denotes the vector of the uncertain parameters of the structural matrices, ϕ_0 is the eigenvector obtained from the baseline eigenvalue problem and $\frac{\partial \phi}{\partial \theta_i}$ are the partial derivatives of the eigenvector evaluated at the nominal solution. In order to compute the undetermined coefficients $\zeta_1(\theta)$ and $\zeta_2(\theta)$, the approximation of the eigenvectors in Eq. 27 is used in the random eigenvalue problem as stated in Eq. 26. This leads to a 2×2 eigenvalue problem of the form

$$\mathbf{K}_R(\theta)Z(\theta) = \lambda(\theta)\mathbf{M}_R(\theta)Z(\theta), \quad (28)$$

where the explicit formulation of the elements of the reduced matrices $\mathbf{K}_R(\theta)$ and $\mathbf{M}_R(\theta)$ can be found in [36].

4.4 Linear Interpolation of Substructure Matrices

Another approach for the simulation of structural matrices by means of a set of previously generated structural matrices consists in the representation of structural matrices as a linear combination of matrices. Hence, instead of taking one simulated matrix for each substructure randomly from the sample pool, these matrices are approximated using two previously generated matrices. In this way, a meta-model for constructing efficiently structural matrices and for solving the corresponding eigenanalysis in a reduced space can be constructed.

4.4.1 Generation of the Sample Pool

First, a set of input points and their associated substructure matrices $\mathbf{K}(\theta)$ and $\mathbf{M}(\theta)$ are generated as discussed in Sect. 4.3, where the n -dimensional vector θ represents a point in the standard normal space. For these simulations and for the nominal solution $\theta^{(0)} = \mathbf{0}$ the classical general eigensolutions have been computed.

$$\mathbf{K}^{(j)}\phi^{(j)} = \mathbf{M}^{(j)}\phi^{(j)}\Lambda^{(j)}, \quad 0 \leq j \leq J. \quad (29)$$

As opposed to the approach in Sect. 4.3, a new simulation is not generated by taking *one* simulation of each substructure from the pool, but by using *two* samples of each substructure.

4.4.2 Approximation of Structural Matrices and Eigenvectors

Any random point $\theta^{(k)}$, $k > J$ lying in a two dimensional subspace spanned by two randomly selected points from the sample pool can be represented as

$$\theta^{(k)} = \xi_1 \theta^{(m)} + \xi_2 \theta^{(n)}, \quad (30)$$

where $\theta^{(m)}$ and $\theta^{(n)}$ are two previously sampled points, i.e. $1 < m, n < J$. The radius R of the point $\theta^{(k)}$, $k > J$ is distributed according to a chi-square distribution and the angle α , which defines the direction in this two-dimensional subspace follows a uniform distribution in the interval $[0, 2\pi]$. Hence, Eq. 30 is written as

$$\theta^{(k)} = (\cos \alpha \mathbf{v}_1 + \sin \alpha \mathbf{v}_3) R, \quad (31)$$

where $\mathbf{v}_1$ is the normalized vector $\theta^{(m)}$ and defines with $\mathbf{v}_3$ an orthonormal basis for the subspace (see Fig. 1).

The associated structural stiffness and mass matrices are now defined as

$$\mathbf{M}(\theta^{(k)}) := \mathbf{M}^{(k)} = \mathbf{M}^{(0)}[1 - \xi_1 - \xi_2] + \xi_1 \mathbf{M}^{(m)} + \xi_2 \mathbf{M}^{(n)}, \quad (32)$$

$$\mathbf{K}(\theta^{(k)}) := \mathbf{K}^{(k)} = \mathbf{K}^{(0)}[1 - \xi_1 - \xi_2] + \xi_1 \mathbf{K}^{(m)} + \xi_2 \mathbf{K}^{(n)}, \quad (33)$$

where $\mathbf{M}^{(m)}$ and $\mathbf{K}^{(m)}$ denote $\mathbf{M}(\theta^{(m)})$ and $\mathbf{K}(\theta^{(m)})$, respectively. Moreover, the following linear approximation makes sense, because it approximates the eigen-solution in the close neighborhood of the established solutions for $\Phi^{(j)}$, $j \in [0, J]$ very well.

$$\Phi(\theta^{(k)}) \approx \tilde{\Phi}^{(k)} = \Phi^{(0)}[1 - \xi_1 - \xi_2] + \xi_1 \Phi^{(m)} + \xi_2 \Phi^{(n)} \quad (34)$$

The quality of the approximation for points $\theta^{(k)}$, which are distant from the two points $\theta^{(m)}$ and $\theta^{(n)}$, must be investigated by numerical tests.

In the following, however, it is not assumed that these values are accurate, but that they establish the correct subspace, in which the vectors $\tilde{\Phi}^{(k)}$ describe the subspace

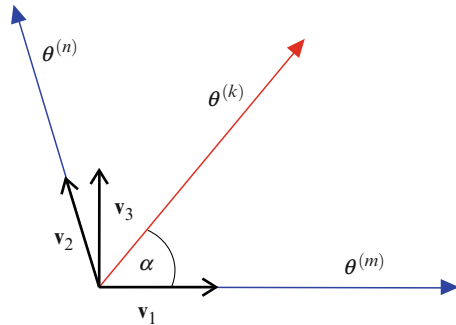


Fig. 1 Representation of $\theta^{(k)}$ by the two samples $\theta^{(m)}$ and $\theta^{(n)}$

spanned by the eigenvectors of the matrices $\mathbf{M}^{(k)}$ and $\mathbf{K}^{(k)}$. This assumption implies that the correct eigenvectors are a linear combination of the vectors in the matrix $\tilde{\boldsymbol{\Phi}}^{(k)}$, which can be represented by

$$\boldsymbol{\Phi}^{(k)} = \tilde{\boldsymbol{\Phi}}^{(k)} \mathbf{Q}^{(k)} \quad (35)$$

This transformation matrix $\mathbf{Q}^{(k)}$ can be determined by the reduced eigensolution

$$\mathbf{K}_R^{(k)} \mathbf{Q}^{(k)} = \mathbf{M}_R^{(k)} \mathbf{Q}^{(k)} \boldsymbol{\Lambda}^{(k)} \quad (36)$$

where the reduced matrices read

$$\mathbf{K}_R^{(k)} = \tilde{\boldsymbol{\Phi}}^{(k)T} \mathbf{K}^{(k)} \tilde{\boldsymbol{\Phi}}^{(k)} \quad (37)$$

$$\mathbf{M}_R^{(k)} = \tilde{\boldsymbol{\Phi}}^{(k)T} \mathbf{M}^{(k)} \tilde{\boldsymbol{\Phi}}^{(k)}. \quad (38)$$

The reduced eigenvalue problem provides also the eigenvalues.

5 Numerical Example

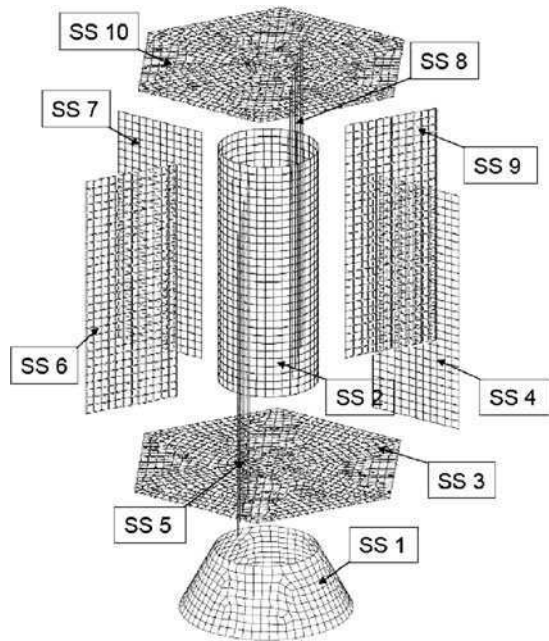
5.1 Problem Statement

As a numerical example a satellite structure is used. This model is divided into the following ten substructures (shown in Fig. 2 [33]):

- Substructure 1: adapter
- Substructure 2: central cylinder
- Substructure 3: lower platform
- Substructures 4–9: panel no. 1–6
- Substructure 10: upper platform

The connections between the substructures are modeled using elastic springs leading to a total number of 187 springs with 3 translational and 3 rotational DOFs. On each of the six panels, concentrated masses are added in order to model some equipment. The uncertainties of the structural parameters, i.e. of spring stiffnesses, concentrated masses, thicknesses, Young's moduli and densities, are considered, where truncated Gaussian distributions with a coefficient of 20% (spring stiffnesses) and 2.5% (concentrated masses, thicknesses, Young's moduli and densities), respectively, are assumed. A fine mesh is adopted to model the build-up structure leading to approximately 36,000 DOFs. This size of the structural matrices makes it possible to explore also the scalability of the proposed algorithms in addition to investigating their accuracy.

Fig. 2 Numerical example [33] split into ten substructures



As a reference solution 1,000 Monte Carlo samples are generated and compared with the four methods presented in the previous section. The quantities of interest are the statistics of the eigenvalues. For the methods that are based on a random combination of substructures, a set of 50 substructure matrices is generated and combined randomly in order to assess the statistical description of the output quantities of interest. In order to capture the randomness with few samples, Latin Hypercube sampling is adopted.

5.2 Results

5.2.1 Karhunen-Loève Expansion of the Substructure Matrices

Figure 3 shows the comparison of the histograms of the first six eigenfrequencies. The second and third mode have closely spaced eigenfrequencies, for which reason only one peak is visible. The mean values of the six eigenfrequencies (apart mode no. 5) can be predicted well using the KL-decomposition of the substructure matrix entries. The standard deviation leads to an underestimation of the reference scatter as obtained with direct MCS of the full model.

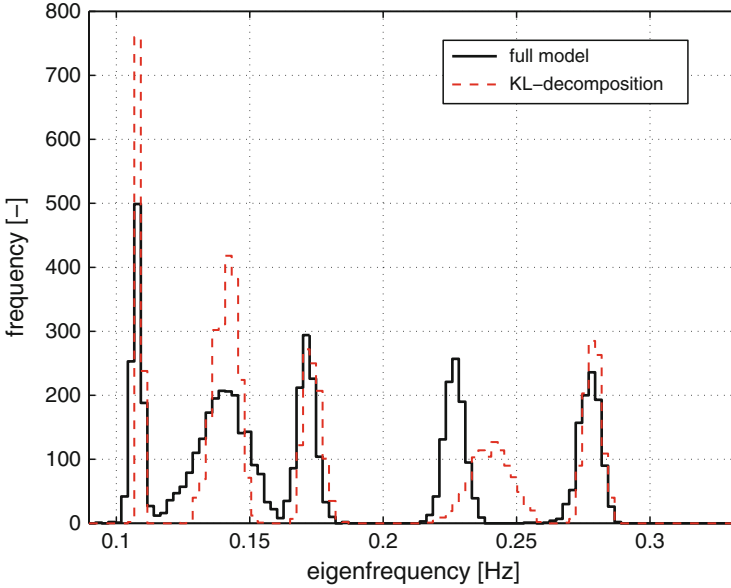


Fig. 3 Histograms of the first six eigenfrequencies obtained with the approach using KL-decomposition (*dashed line*) and the reference solution (*solid line*) obtained with 1,000 simulations

5.2.2 Random Combination of Substructures

As a next step, the results obtained by randomly combining substructure matrices (see Sect. 4.3) are compared with the reference solution. The results in Fig. 4 reveal a good agreement with the reference statistics of the eigenfrequencies. Hence, the random combination of substructures introduces additional variability with respect to the variability given by the sample pool of substructure matrices. This leads to an accurate representation of the statistics of the eigenfrequencies while only a small number of assemblies of substructure matrices with associated eigensolution have to be performed.

5.2.3 Stochastic Reduced Basis Approximation (SRBA)

The results shown in Fig. 5 are also obtained by randomly combining substructure matrices as described in the previous case. However, for this analysis the generation of the sample pool is performed with a decreased computational effort since the modal parameters of the substructure matrices are approximated as described in Sect. 4.3.1. The small variability of the Young's modulus and density combined with the fact that no mode switching occurs at the substructure level leads to a high accuracy of the employed approach. As observed in the previous analysis, the histograms of the first six eigenfrequencies show good agreement.

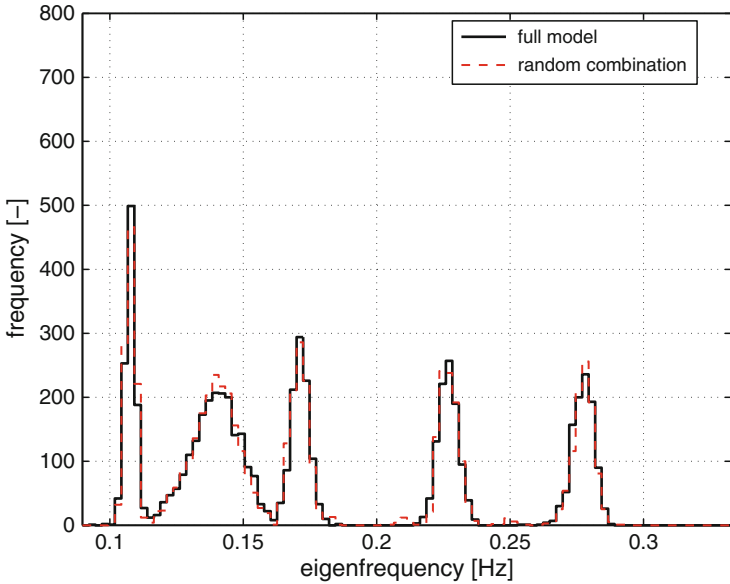


Fig. 4 Histograms of the first six eigenfrequencies obtained with the approach using the random combination of substructures (*dashed line*) and the reference solution (*solid line*) based on 1,000 simulations

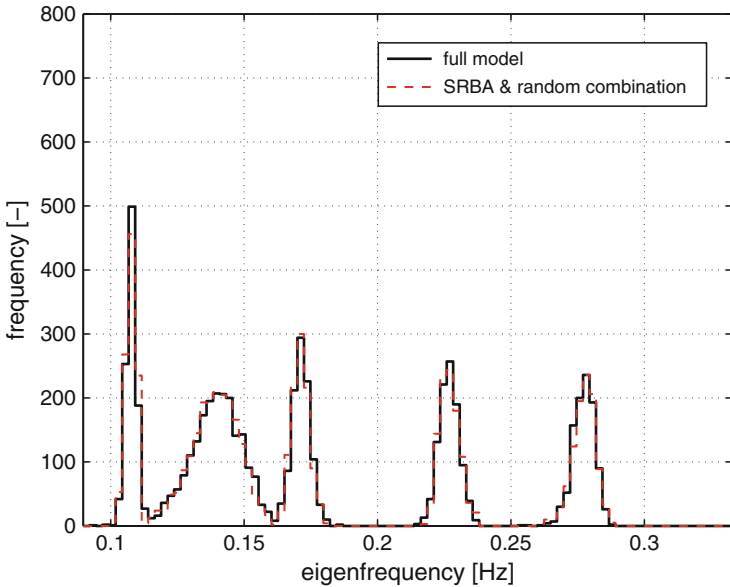


Fig. 5 Histograms of the first six eigenfrequencies obtained with the approach using SRBA and the random combination of substructures (*dashed line*) and the reference solution (*solid line*) obtained with 1,000 simulations

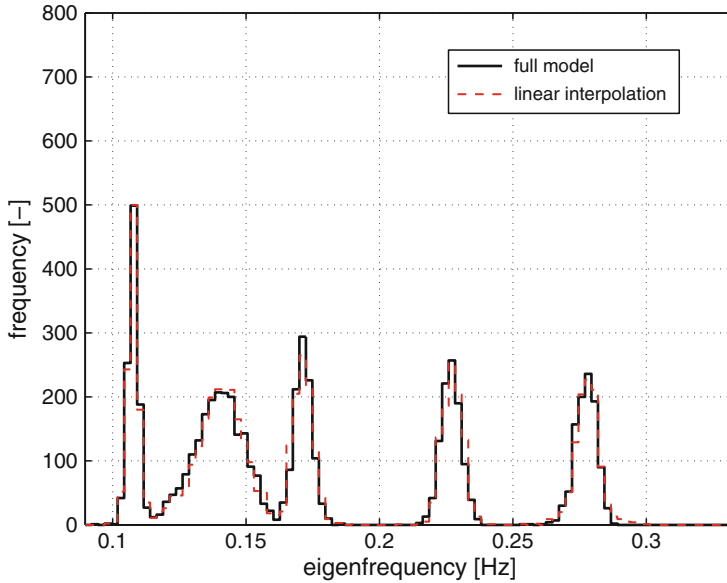


Fig. 6 Histograms of the first six eigenfrequencies using the approach applying linear interpolation of the structural matrices (*dashed line*) and the reference solution (*solid line*) obtained with 1,000 simulations

5.2.4 Linear Interpolation of Substructure Matrices

The last comparison is devoted to the approach involving a linear interpolation of the substructure matrices. The results as shown in Fig. 6 are obtained by applying the method as presented in Sect. 4.4. The two supporting points are selected randomly from a set of 50 previously simulated input vectors and corresponding structural matrices and matrices of eigenvectors. The new simulation is enforced to lie in this subspace, where the radius is a realization of a chi-square distribution and the angle is sampled from a uniform distribution in the interval $[0, 2\pi]$.

The figure reveals that high accuracy can be obtained with the employed approximation and that the statistics of the reference solution can be reproduced. The good agreement of the second peak which involves second and third eigenfrequencies shows that this method has no apparent difficulty with the effects of mode switching. As opposed to the other methods presented, this approach is applied on the full structure and not on the components where no mode switching occurs. In addition, the results also show that the tails of the histograms obtained with direct MCS of the full model and the approximation method show good agreement.

5.2.5 Computational Efforts

Finally, the savings in the computational costs are analyzed. Figure 7 shows the computational costs for the FRF-computation, which shows remarkable time

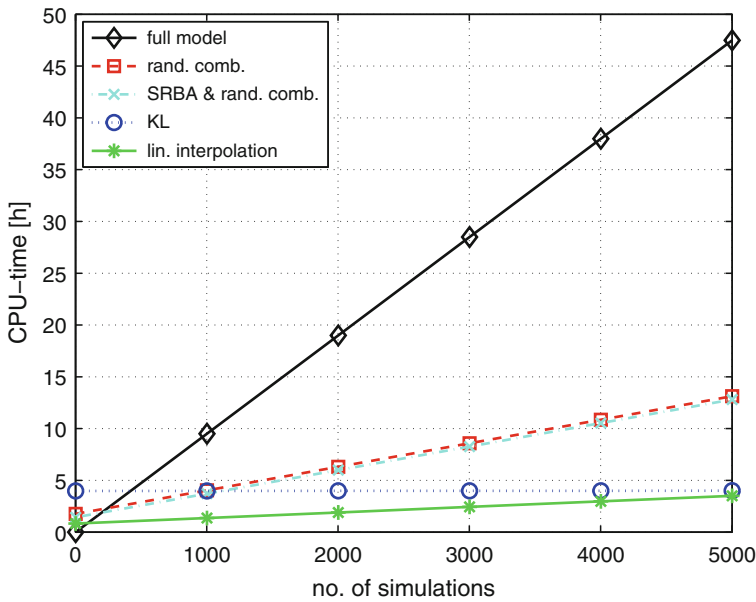


Fig. 7 Comparison of the computational costs of the adopted approaches as a function of the number of simulations

savings for the methods where efficient uncertainty propagation methods are combined with substructuring. It can be observed that the three methods described in the previous section do not start from the origin due to the necessary preparatory work for the analysis which comprises the generation of a pool of substructure matrices, the determination of the parameters of the KL-expansion or the computation of the derivatives of the eigenvectors. But the time for one simulation is considerably smaller than the time needed for one run of the full model.

6 Conclusions

In this manuscript the well established and most commonly used deterministic reduction techniques are discussed. Special attention was given to the widely used Guyan reduction and the Craig-Bampton method. Methods for the quantification and efficient propagation of uncertainty are summarized and approaches which combine stochastic analysis with reduction techniques are reviewed. The use of deterministic reduction techniques within a stochastic analysis is especially advantageous in view of the computational savings, but also regarding the additional possibilities for uncertainty consideration.

Two new procedures which combine the advantages of efficient simulation strategies and of substructuring are proposed and verified by means of a numerical example. The results show that high accuracy can be achieved while cutting down

the computational costs remarkably. Future developments will be oriented towards the parallelization of the approaches which offers an additional possibility to decrease the wall clock time even further and to make probabilistic analysis efficient also for highly complex dynamical systems.

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References

1. Allemang, R.J., Brown, D.: A correlation coefficient for modal vector analysis. *Proceedings of the 1st IMAC*, pp. 110–116, Orlando (1982)
2. Au, S.-K., Beck, J.L.: Estimation of small failure probabilities in high dimensions by subset simulation. *Probabilist. Eng. Mech.* **16**(4), 263–277 (2001)
3. Bah, M.T., Nair, P.B., Bhaskar, A., Keane, A.J.: Stochastic component mode synthesis. In: 44th AIAA/ASCE/ASME/AHS Structures, Structural Dynamics and Material Conference, AIAA-2003-1750, Norfolk (2003)
4. Benfield, W.A., Hruda, R.F.: Vibration analysis of structures by component mode substitution. *AIAA J.* **9**(7), 1255–1261 (1971)
5. Bouhaddi, N., Fillard, R.: Model reduction by a simplified variant of dynamics condensation. *J. Sound Vib.* **191**(2), 233–250 (1996)
6. Box, G.E.P., Wilson, K.B.: On the experimental attainment of optimum conditions. *J. Roy. Statist. Soc.* **13**(1), 1–45 (1951)
7. Castanier, M.P., Tan, Y.-C., Pierre, C.: Characteristic constraint modes for component mode synthesis. *AIAA J.* **39**(6), 1182–1187 (2001)
8. Cornell, J.A., Khuri, A.I.: *Response Surfaces: Designs and Analyses*, Volume 81 of *Statistics: Textbooks and Monographs*. Marcel Dekker, New York (1987)
9. Craig, R.R., Jr.: A brief tutorial on substructure analysis and testing. In: 18th International modal analysis conference, San Antonio (2000)
10. Craig, R.R., Jr.: Coupling of substructures for dynamic analyses: An overview. *AIAA Paper No. 2000-1573*, AIAA Dynamics Specialists Conference, Atlanta (2000)
11. Craig, R.R., Jr., Bampton, M.M.: Coupling of substructures for dynamical analysis. *AIAA J.* **6**(7), 1313–1319 (1968)
12. Craig, R.R., Jr., Chang, C.-J.: Free-interface methods of substructure coupling for dynamic analysis. *AIAA J.* **14**(11), 1633–1635 (1976)
13. Craig, R.R., Jr., Chang, C.-J.: On the use of attachment modes in substructure coupling for dynamic analysis. In: 18th AIAA/ASME Structures, Structural Dynamics and Material Conference, San Diego (1977)
14. De Kraker, A., Van Campen, D.A.: Rubin's CMS reduction method for general space-state models. *Comput. Struct.* **58**(3), 597–606 (1996)
15. Eldred, M.S., Giunta, A.A., Castro, J.P.: Uncertainty quantification using response surface approximations. In: *Proceedings of the 9th ASCE Joint Specialty Conference on Probabilistic Mechanics and Structural Reliability*, Albuquerque (2007)
16. Fishman, G.S.: *Monte Carlo: Concepts, Algorithms, Applications*. Springer, New York (1996)
17. Ghanem, R., Kruger, R.: Numerical solution of spectral stochastic finite element systems. *Comput. Method. Appl. Mech. Eng.* **129**, 289–303 (1996)
18. Ghanem, R., Spanos, P.: *Stochastic Finite Elements: A Spectral Approach*. Springer, New York (1991)
19. Ghanem, R., Ghosh, D., Red-Horse, J.: Analysis of eigenvalues and modal interaction of stochastic systems. *AIAA J.* **43**(10), 2196–2201 (2005)

20. Giunta, A., Watson, L., Koehler, J.: A comparison of approximation modeling techniques: Polynomial versus interpolating models. In: 7th AIAA/USAF/NASA/ISSMO Symposium on Multidisciplinary Analysis and Optimization, St. Louis (1998)
21. Guyan, R.: Reduction of stiffness and mass matrices. *AIAA J.* **3**(2), 380 (1965)
22. Hintz, R.M.: Analytical methods in component modal synthesis. *AIAA J.* **13**(8), 1007–1016 (1975)
23. Hurty, W.C.: Dynamic analysis of structural systems using component modes. *AIAA J.* **3**(4), 678–685 (1965)
24. Jin, R., Chen, W., Simpson, T.W.: Comparative studies of metamodeling techniques under multiple modelling criteria. *Struct. Multidiscip. Optim.* **23**(1), 1–13 (2001)
25. Johnson, E.A., Wojtkiewicz, S.F., Bergman, L.A., Spencer, B.F., Jr.: Observations with regard to massively parallel computation for Monte Carlo simulation of stochastic dynamical systems. *Int. J. Nonlinear Mech.* **32**(4), 721–734 (1997)
26. Johnson, E.A., Proppe, C., Spencer, B.F. Jr., Bergman, L., Székely, G.S., Schuëller, G.I.: Parallel processing in computational stochastic dynamics. *Probabilist. Eng. Mech.* **18**(1), 37–60 (2003)
27. Keane, A.J., Nair, P.B.: *Computational Approaches for Aerospace Design*. Wiley, New York (2005)
28. Kleiber, M., Hien, T.D.: *The stochastic finite element method: basic perturbation technique and computer implementation*. Wiley, New York (1992)
29. Koutsourelakis, P.S., Pradlwarter, H.J., Schuëller, G.I.: Reliability of structures in high dimensions, part I: Algorithms and applications. *Probabilist. Eng. Mech.* **19**(4), 409–417 (2004)
30. Mace, B.R., Shorter, P.J.: A local modal/perturbational method for estimating frequency response statistics of built-up structures with uncertain properties. *J. Sound Vib.* **242**(5), 793–811 (2001)
31. MacNeal, R.H.: A hybrid method of component mode synthesis. *Comput. Struct.* **1**(4), 581–601 (1971)
32. Mayer, M.: *Die Sicherheit der Bauwerke und ihre Berechnung nach Grenzkraften anstatt nach zulässigen Spannungen*. Springer, Berlin (1926)
33. MCS Software Corporation: *MCS Nastran 2001: Superelement User's Guide*, Santa Ana, CA 92707 USA (2004)
34. Myers, R.H., Montgomery, D.C.: *Response Surface Methodology: Process and Product Optimization Using Designed Experiments*. Wiley, New York (2002)
35. Nair, P.B., Keane, A.J.: Stochastic reduced basis methods. *AIAA J.* **40**(8), 1653–1664 (2002)
36. Nair, P.B., Keane, A.J.: An approximate solution scheme for the algebraic random eigenvalue problem. *J. Sound Vib.* **260**(1), 45–65 (2003)
37. Ohayon, R., Sampaio, R., Soize, C.: Dynamic substructuring of damped structures using singular value decomposition. *J. Appl. Mech.* **64**, 292–298 (1997)
38. Pellissetti, M.: Parallel processing in structural reliability. *J. Struct. Eng. Mech.* **32**, 95–126 (2009)
39. Pichler, L., Pradlwarter, H.J., Schuëller, G.I.: A mode-based meta-model for the frequency response functions of uncertain structural systems. *Comput. Struct.* **87**(5–6), 332–341 (2009)
40. Pradlwarter, H.J., Schuëller, G.I.: Stochastic response evaluation of large FE-models with nonlinear hysteretic elements by complex mode synthesis. In: Spanos, P.D. (ed.) *Proceedings of the Third International Conference on Computational Stochastic Mechanics*, pp. 551–558, Santorini, Greece. A.A. Balkema, Rotterdam (1999)
41. Pradlwarter, H.J., Schuëller, G.I., Székely, G.S.: Random eigenvalue problems for large systems. *Comput. Struct.* **80**, 2415–2424 (2002)
42. Rahman, S.: Stochastic dynamic systems with complex-valued eigensolutions. *Int. J. Num. Meth. Eng.* **71**, 963–986 (2007)
43. Rubin, S.: Improved component-mode representation for structural dynamic analysis. *AIAA J.* **13**(8), 995–1006 (1975)
44. Sachdeva, S.K., Nair, P.B., Keane, A.J.: Comparative study of projection schemes for stochastic finite element analysis. *Comput. Method. Appl. Mech. Eng.* **195**(19–22), 2371–2392 (2006)

45. Sacks, J., Schiller, S.B., Welch, W.J.: Design for computer experiment. *Technometrics* **31**(1), 41–47 (1989)
46. Schuëller, G.I.: Structural reliability – recent advances – Freudenthal Lecture. In: Shiraishi, N., Shinozuka, M., Wen, Y.K. (eds.) *Proceedings of the 7th International Conference on Structural Safety and Reliability (ICOSSAR'97)*, pp. 3–35, Kyoto, Japan. A.A. Balkema, Rotterdam (1998)
47. Schuëller, G.I.: Developments in stochastic structural mechanics. *Arch. Appl. Mech.* **75**, 755–773 (2006)
48. Schuëller, G.I.: On the treatment of uncertainties in structural mechanics and analysis (based on a Plenary Keynote Lecture presented at the Third M.I.T. Conference on Computational Fluid and Solid Mechanics), Boston, USA. *Comput. Struct.* **85**(5–6): 235–243 (2007)
49. Schuëller, G.I. (ed.): A state-of-the-art report on computational stochastic mechanics. *J. Probabilist. Eng. Mech.* **12**(4), 197–321 (1997)
50. Schuëller, G.I., Stix, R.: A critical appraisal of methods to determine failure probabilities. *J. Struct. Safety*, **4**, 293–309 (1987)
51. Soize, C.: A nonparametric model of random uncertainties for reduced matrix models in structural dynamics. *Probabilist. Eng. Mech.* **15**, 277–294 (2000)
52. Soize, C.: Maximum entropy approach for modeling random uncertainties in transient elastodynamics. *J. Acoust. Soc. Am.* **109**, 1979–1996 (2001)
53. Székely, G.S., Schuëller, G.I.: Computational procedure for a fast calculation of eigenvectors and eigenvalues of structures with random properties. *Comput. Method. Appl. Mech. Eng.* **191**(8–10): 799–816 (2001)
54. Székely, G.S., Teichert, W.H., Brenner, C.E., Pradlwarter, H.J., Klein, M., Schuëller, G.I.: Practical procedures for reliability estimation of spacecraft structures and their components. *AIAA J.* **36**(8), 1509–1515 (1998)
55. Vasta, M., Schuëller, G.I.: Phase space reduction in stochastic dynamics. *J. Eng. Mech.* **126**(6), 626–632 (2000)
56. Verhoosel, C.A., Gutiérrez, M.A., Hulsdoff, S.J.: Iterative solution of the random eigenvalue problem with application to spectral stochastic finite element analysis. *Int. J. Numer. Method. Eng.* **68**, 401–424 (2006)

Resonant Damping of Flexible Structures Under Random Excitation

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Abstract Structural vibrations are often dominated by resonant response, and increased efficiency in the damping of these vibrations can often be attained by using the resonant properties of these modes. A typical example is the ‘tuned mass absorber’. While the original design procedure was based on properties of the frequency response graph, it has recently been demonstrated that the problem can be generalized and solved by use of the complex root-locus properties. The basic principle in this formulation is the introduction of a resonant force with frequency tuning that results in splitting the original resonant mode into two modes with equal damping ratio. Here the basic principle of resonant absorbers is presented in concise form and generalized in two ways. First the principle of resonant absorbers is presented in a general form in terms of sensors and actuators, recording the motion and imposing appropriate forces, respectively. A general design procedure is developed for resonant displacement and acceleration feedback, respectively, based on a combination of ‘equal modal damping’ and approximately equal response amplitudes of the two modes. This leads to explicit design formulae for the parameters of the control system. The optimal calibration leads to a plateau of near-equal amplification in a frequency interval around the original natural frequency. In multi-degree-of-freedom structures the sensor picks up the total deformation from all the modes that are active at the location of the sensor, and the actuator force acts on all these modes. A quasi-static correction is developed that identifies the part of the sensor signal associated with the mode to be damped and the reduced effect of the actuator on this mode. This correction takes the form of explicit modifications of the formulae for the optimal control parameters, while retaining the original format. The efficiency of resonant damping is illustrated by application to a benchmark example for stochastic wind load on a high-rise building. It is demonstrated that the present resonant damping technique based on a single collocated sensor is competitive with more heavily instrumented configurations using the classic LQG technique.

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1 Introduction

There are many situations within structural dynamics, in which there is a need for limiting undesirable oscillations at one or more resonance frequencies, e.g. earthquake response, bridge vibrations, space structures and instruments. In the structural dynamics community this has typically been dealt with by using the concepts from the tuned mass absorber, described e.g. by Den Hartog [1]. The idea is to mount an additional mass with suitable stiffness and damping to introduce a resonant relative motion and to optimize the damping of this relative motion. The traditional design procedure is based on the frequency response curve of an idealized single degree of freedom structure. First the frequency is tuned to balance the magnitude of the resonances of the two modes generated by the combined motion of the idealized structure and the suspended mass, and then the damping is chosen to provide suitable extraction of energy without eliminating the relative motion.

Within the area of structural control the effect is typically obtained by actuators that are controlled by applying a suitable algorithm to signals obtained from sensors on the structure. In this setting the resonant control is typically obtained by a suitable filter that introduces an additional resonance, represented by an additional pole in the complex frequency plane. Common design procedures are based on placement of the complex poles to obtain maximum modal damping, and the intensity of the actuator force is controlled via a gain parameter [2]. In the case of resonant control identification of a suitable format containing a gain parameter is a central part of the design problem.

A common characteristic of most of the work on resonant control from a control theory perspective is its focus of the location of the poles in the complex plane. However, optimal response characteristics also require optimal coupling between control and structure, and may also involve the load – properties that do not show up directly in the root locus diagram. In a recent analysis by Krenk [3] the original response curve design format of the tuned mass damper was recast into root locus format, and it was demonstrated that a key to the efficiency of the classic design procedure is that the two modes generated by interference have identical modal damping ratio. This turns out to be a convenient starting point for the design of resonant dampers. The first step is to select the resonant frequency of the device to produce the ‘equal modal damping split’. The remaining parameters should then be chosen to provide a level plateau with suitable damping. This in turn implies that the two involved resonance frequencies are not too closely spaced, as this would produce undesirable constructive interference. This was demonstrated for the tuned mass absorber in [3] and extended to a more general format for resonant damping in terms of sensors and actuators in [4].

In the case of a flexible structure the motion monitored by the sensor will contain components that are not associated with the mode to be controlled. This problem has been addressed in connection with a tuned mass absorber on a multi-degree-of-freedom structure by Ozer and Royston [5,6], who included the additional degree of freedom by use of the Sherman-Morrison formula and established an analogy with the single degree of freedom system and the design procedure of Den Hartog. In its

general form the procedure is rather cumbersome, involving an iteration process. In this paper a simple explicit correction is developed for control of the low-frequency modes. The effect of the higher frequency modes is taken into account by including the corresponding quasi-static terms in the two equations for the simple control problem. The procedure and its efficiency are demonstrated by application to a high-rise building under wind load.

2 Resonant Response Format

The control problem is illustrated in Fig. 1. The structure is represented by a mass m with displacement $x(t)$. The external load on the structure is represented by the force $F(t)$. The control is accomplished via a sensor signal $y(t)$ used to control an additional force $F_c(t)$. Figure 1a illustrates the case of a flexible structure, with the first mode being represented by the mode shape vector $\mathbf{u}_1$ and the modal amplitude x_1 . A single-degree-of-freedom representation is obtained, if this mode is considered without including interaction with higher modes. This case will be considered first in Sects. 3 and 4. In some situations it is important to account for the fact that the modal response $\mathbf{u}_1 x_1(t)$ only constitutes part of the total motion. In this case improved damping properties may be obtained by correcting for the contributions from higher frequency modes as discussed in Sect. 5.

The analysis and design of the system will be carried out in the frequency domain. For this purpose the time dependence is represented via the complex exponential function $\exp(i\omega t)$, where ω is the angular frequency, that will typically be complex valued, corresponding to attenuated response. The controller force F_c is obtained from the control variable y , which in turn is obtained by filtering the response x . This can be expressed in the general frequency format

$$\begin{aligned} G_{xx}(\omega) x + G_{xy}(\omega) y &= F/m, \\ G_{yx}(\omega) x + G_{yy}(\omega) y &= 0, \end{aligned} \quad (1)$$

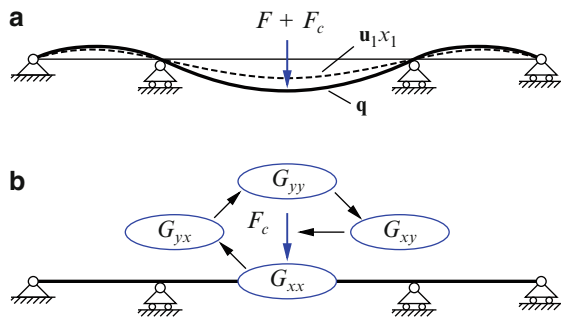


Fig. 1 Structure with sensor and control force: (a) Flexible MDOF structure and (b) control loop

where F is the external force, and m is the structural mass. $G_{xx}, G_{xy}, \dots$ are frequency dependent transfer functions, illustrated in Fig. 1b.

The quality of the control is associated with its ability to limit the response of the structure around the resonance frequency ω_s . The response x follows from elimination of the control variable y between the equations in (1), whereby

$$x = \frac{G_{yy}}{G_{xx}G_{yy} - G_{xy}G_{yx}} \frac{F}{m}. \quad (2)$$

Clearly, the magnitude of the control force F_c needed to reduce the resonant response is also of importance. It follows from (1) in the form

$$F_c = -m G_{xy} y = \frac{G_{xy}G_{yx}}{G_{xx}G_{yy} - G_{xy}G_{yx}} F. \quad (3)$$

It is seen that the frequency functions G_{xy} and G_{yx} only influence the response x and the control force F_c through the product $G_{xy}G_{yx}$. Interchange of G_{xy} with G_{yx} may lead to a redefinition of the control variable y , but leaves the structural response x and the control force F_c unchanged.

2.1 Structure and Resonator Representation

For a structure without internal damping the natural angular frequency ω_s is given in terms of the stiffness k and mass m as

$$\omega_s^2 = k/m. \quad (4)$$

Thus, the free response of the structure is governed by the frequency function

$$G_{xx}(\omega) = \omega_s^2 - \omega^2. \quad (5)$$

In this formulation the effect of structural damping has been omitted because it is usually small for structures needing additional damping devices and furthermore it complicates the analysis considerably. A simple combination format for structural and controller induced damping has been obtained in [7] for the tuned mass absorber, corresponding to the case of acceleration feedback in the present context.

The controller consists of an oscillator defined by the second order filter function

$$G_{yy}(\omega) = \omega_c^2 - \omega^2 + 2i\zeta_c\omega_c\omega, \quad (6)$$

where ω_c represents a characteristic angular frequency, and the non-dimensional parameter ζ_c defines the bandwidth of the filter. In the case of a tuned mass vibration absorber ω_c and ζ_c are simply the undamped natural frequency and the damping ratio of the suspended mass system if connected to a rigid support [1, 3].

2.2 Feedback Filters

It is desirable that the control force ratio F_c/F vanishes at zero and infinite frequencies. Within the present format this implies that the product $G_{xy}G_{yx}$ can contain cubic, quadratic and linear terms, but neither a constant or a quartic term. This condition is satisfied by generalized acceleration and displacement feedback. In the following these two feedback forms are treated separately, using general expressions containing the terms from both formulations.

In the generalized acceleration feedback format the motion is monitored via the acceleration, e.g. by an accelerometer, corresponding to the feedback function

$$G_{yx}(\omega) = \omega^2. \quad (7)$$

The product $G_{xy}G_{yx}$ is limited to a cubic polynomial. The actuator is controlled by a feedback function with an imaginary part to enable tuning of the phase,

$$G_{xy}(\omega) = \alpha \omega_c^2 + \beta 2i \zeta_c \omega_c \omega, \quad (8)$$

with gain parameters α and β . The design problem consists in the determination of optimal combinations of the frequency ratio ω_c/ω_s , the bandwidth parameter ζ_c , and the two gain parameters α and β .

In the case of displacement feedback the displacement is sampled directly, corresponding to a constant frequency function

$$G_{yx}(\omega) = \omega_c^2, \quad (9)$$

where the frequency ω_c from the resonant controller has been used for dimensional reasons. The actuator frequency function G_{xy} combines a quadratic and a linear term in ω in order to enable tuning of the phase,

$$G_{xy}(\omega) = \alpha \omega^2 + \gamma 2i \zeta_c \omega_s \omega. \quad (10)$$

Here the structure reference frequency ω_s has been used, and two non-dimensional gain factors α and γ have been introduced.

These cases of acceleration feedback and displacement feedback can be combined via the product formula

$$G_{xy}(\omega)G_{yx}(\omega) = \alpha \omega_c^2 \omega^2 + 2i \zeta_c \omega_c \omega (\beta \omega^2 + \gamma \omega_c \omega_s), \quad (11)$$

which covers both cases by setting $\gamma = 0$ and $\beta = 0$, respectively. The denominator then follows from combining the response functions G_{xx} and G_{yy} from (5) and (6) with this expression:

$$\begin{aligned} G_{xx}G_{yy} - G_{xy}G_{yx} = & \omega^4 - [\omega_s^2 + (1 + \alpha)\omega_c^2] \omega^2 + \omega_s^2 \omega_c^2 \\ & - 2i \zeta_c \omega_c \omega [(1 + \beta)\omega^2 - \omega_s^2 + \gamma \omega_s \omega_c]. \end{aligned} \quad (12)$$

The roots of this quartic polynomial determine the complex resonance frequencies, and thereby the damping properties of the resonant modes.

3 Root Locus Diagram

The following root locus analysis generalizes that of the tuned mass absorber presented in [3]. Let the four roots of the characteristic equation be denoted $\omega_1, \dots, \omega_4$, and assume that there is a parameter combination for which ω_1 and ω_2 lie in the first quadrant. The corresponding modes will then have equal damping ratio, if ω_1 and ω_2 lie on the same line containing the origin of the complex plane as illustrated in Fig. 2. This implies that they are inverse points in the complex plane with respect to some real-valued frequency ω_0 , i.e.

$$\frac{\omega_2}{\omega_0} = \frac{\omega_0}{\omega_1^*}, \quad (13)$$

where ω_1^* denotes the conjugate of ω_1 . The reciprocal relation between ω/ω_0 and ω_0/ω suggests the following format of the characteristic equation,

$$\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)^2 - 4i\lambda\xi \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right) - 4\xi^2 = 0. \quad (14)$$

This equation contains two coefficients, and it turns out to be convenient to express these in terms of the real-valued parameters ξ and λ as shown. The corresponding polynomial form follows from multiplication with $(\omega_0\omega)^2$,

$$\omega^4 - (2 + 4\xi^2)\omega_0^2\omega^2 + \omega_0^4 - 4i\lambda\xi\omega_0\omega(\omega^2 - \omega_0^2) = 0. \quad (15)$$

It is seen that the reference frequency ω_0 is defined by the constant term of the normalized characteristic polynomial. The special property of equally damped modes as expressed by the inverse root relation (13) is equivalent to imposing a balance between the cubic and linear terms, whereby they cancel at the frequency $\omega = \pm\omega_0$.

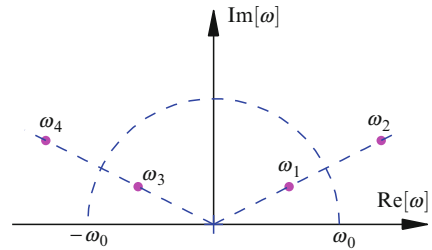


Fig. 2 Complex roots ω_1, ω_2 and ω_3, ω_4 as inverse points of the circle $|\omega| = \omega_0$

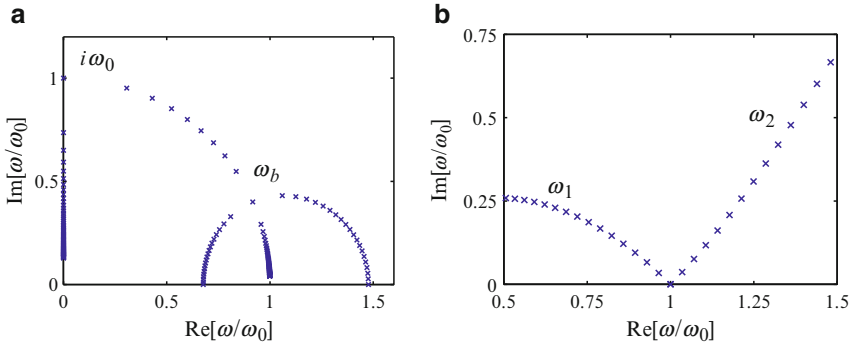


Fig. 3 Root locus diagram: (a) $\xi = 0.4$, $\lambda = 0, 0.05, 0.10, \dots$; (b) $\xi = 0, 0.05, 0.10, \dots$, $\lambda = \frac{1}{2}\sqrt{2}$

The root locus diagram of the quartic equation (15) is easily analyzed by reverting to the quadratic format (14). This format permits determination of the expression inside the parenthesis, and ω then follows from solving a quadratic equation, see e.g. [3, 4] for details. The results are illustrated in Fig. 3. For $\lambda = 0$ there is no damping, and the natural frequencies ω_1 and ω_2 appear as points on the positive real axis. When increasing λ for a fixed value of the parameter $\xi < 1$, the roots move into the complex plane as illustrated in Fig. 3a. It follows from the inverse point property, illustrated in Fig. 2, that the two roots ω_1 and ω_2 have equal argument, corresponding to equal damping of the corresponding free modes. This situation changes at the bifurcation point ω_b , which is reached for $\lambda = 1$. The roots then branch off along a circle with radius $|\omega| = \omega_0$. One branch reaches a branch point on the imaginary axis and the other follows the circle towards the real axis.

The parameter values of interest in connection with control and damping of structures correspond to the first part where the roots are inverse points in the circle, i.e. the parameter interval $0 < \lambda \leq 1$. In this parameter interval the two complex roots have the form

$$\omega_{1,2} = |\omega_{1,2}|(\sqrt{1-\zeta^2} + i\zeta), \quad (16)$$

where ζ is the common damping ratio of the two modes. A simple expression for the modal damping ratio ζ in terms of the system parameters λ and ξ can be obtained by using the fact that the coefficient of the cubic term in (15) is the sum of the four roots $\omega_1, \dots, \omega_4$. When using the special properties of the four roots illustrated in Fig. 2, the following expression is obtained for the damping ratio

$$\zeta = \frac{\lambda \xi}{\frac{1}{2}\left(\frac{|\omega_1|}{\omega_0} + \frac{\omega_0}{|\omega_1|}\right)} \simeq \lambda \xi, \quad (17)$$

where the latter approximation is valid for small values of ξ , typical of practice.

It may appear desirable to choose λ corresponding to the bifurcation point, as this introduces maximum damping in the modes. However, this also implies identical

angular frequency of the modes and thereby leads to constructive interference of the two modes. As it turns out, an optimal balance between the obtained damping and a suitable separation of the vibration frequencies leads to the parameter value $\lambda = \frac{1}{2}\sqrt{2}$. The local part of the corresponding root locus diagram is illustrated in Fig. 3b. The two roots ω_1 and ω_2 move into the complex plane along curves forming an angle of $\pm 45^\circ$ with the real axis, leading to a modal damping ratio of

$$\zeta \simeq \frac{1}{2}\sqrt{2}\xi. \quad (18)$$

The following task is to express the optimal properties in terms of the filter parameters and to optimize the corresponding response amplitudes.

4 Optimal Parameters

The root locus analysis of the previous section has identified a characteristic angular frequency ω_0 and determined a condition for equal modal damping in terms of ω_0 . The characteristic frequency is determined from the constant term in the polynomial (12),

$$\omega_0^2 = \omega_s \omega_c. \quad (19)$$

The condition of equal modal damping consists of the vanishing of the imaginary part of the denominator (12), when evaluated at the reference frequency $\omega = \omega_0$. This gives the equation

$$\omega_s^2 - \gamma \omega_s \omega_c = (1 + \beta) \omega_0^2. \quad (20)$$

Substitution of the reference frequency ω_0 from (19) then gives the controller frequency as

$$\omega_c = \frac{\omega_s}{1 + \beta + \gamma} \quad (21)$$

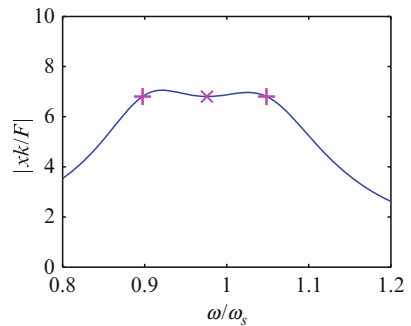


Fig. 4 Equal displacement amplification at three frequencies

Table 1 Optimal parameters for resonant damping in terms of β or γ

Parameter	Acceleration feedback	Displacement feedback
ω_c	$\omega_s/(1 + \beta)$	$\omega_s/(1 + \gamma)$
ω_0^2	$\omega_c \omega_s$	$\omega_c \omega_s$
α	β	$\gamma(1 - \gamma)$
ζ_c^2	$\frac{1}{2} \frac{\beta}{1 + \beta}$	$\frac{\gamma}{4} \frac{2 + \gamma}{1 + \gamma}$

and the reference frequency by

$$\omega_0^2 = \frac{\omega_s^2}{1 + \beta + \gamma}. \quad (22)$$

These expressions are included in Table 1, containing the full set of optimal parameters for the two cases.

The principle for determination of the optimal value of the remaining two parameters α and ζ_c is quite simple, although the computations are a bit elaborate. Both parameters are determined from properties of the frequency response (2). An optimal response curve is illustrated in Fig. 4. It turns out that for both the cases under consideration the frequency response curve for $|xk/F|$ has two points at frequencies ω_A and ω_B , marked by the symbol $+$, where the amplitude is independent of the control damping ratio parameter ζ_c . The parameter α is determined by the condition that the amplitude $x(\omega)k/F$ at the two frequencies ω_A and ω_B are equal. This determines the parameter α as

$$\alpha = \beta, \quad \gamma = 0 \quad (23)$$

in the case of acceleration feedback, and

$$\alpha = \gamma(1 - \gamma), \quad \beta = 0 \quad (24)$$

in the case of displacement feedback. This principle has been used by Den Hartog [1] to determine the frequency ratio of a tuned mass absorber, which is a special case of acceleration feedback. In [3] it was demonstrated that for the tuned mass absorber, this criterion is equivalent to the equal modal damping criterion discussed in Sect. 3. In the general case(s) considered here, the equal modal damping criterion determines the frequency ω_c of the controller, while the equal amplification criterion then determines the gain parameter α .

The idea of a resonance damper is that the amplitude should be limited around the resonance frequencies. A particularly simple and robust way of obtaining near-uniform amplification in an interval around the resonance frequencies is to require the amplification at the reference frequency ω_0 , marked by the symbol $\times$ in Fig. 4, to be equal to that at ω_A and ω_B , [3]. This condition is used to determine the remaining

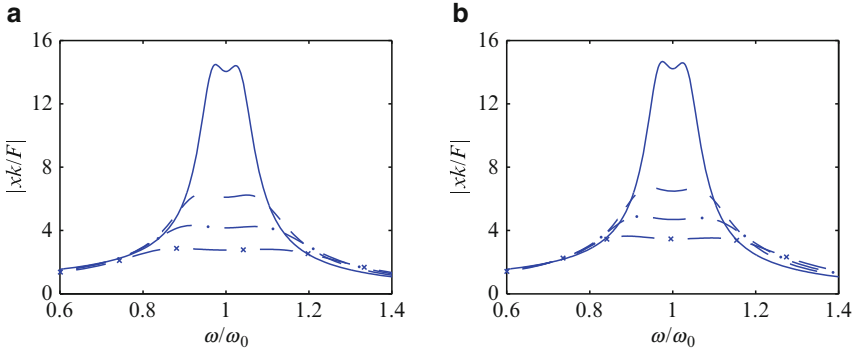


Fig. 5 Frequency response amplitude: $\beta, \gamma = 0.01$ (—), 0.05 (---), 0.1 (- · -) and 0.2 (- × -). (a) Acceleration feedback, (b) displacement feedback

parameter ζ_c . For acceleration feedback this leads to

$$\zeta_c^2 = \frac{1}{2} \frac{\beta}{1 + \beta}, \quad \gamma = 0 \quad (25)$$

while displacement feedback gives

$$\zeta_c^2 = \frac{\gamma}{4} \frac{2 + \gamma}{1 + \gamma}, \quad \beta = 0. \quad (26)$$

These optimal parameters are included in Table 1.

The response characteristics of the two types of devices are illustrated in Fig. 5. They are quite similar with maximum values

$$\left| \frac{x k}{F} \right|_{\omega_0} \simeq \sqrt{2/\beta} \quad \text{and} \quad \left| \frac{x k}{F} \right|_{\omega_0} \simeq \sqrt{2/\gamma} \quad (27)$$

for the two cases.

The present parameter calibration has the interesting property that each of the equally damped modes have approximately half of the damping ratio ζ_c introduced in the controller. This follows from the generic result (17), when applied to the present calibrated filter combination,

$$\zeta \simeq \frac{1}{2} \zeta_c \frac{\omega_c}{\omega_0} \simeq \frac{1}{2} \zeta_c. \quad (28)$$

This approximate result, initially derived for the tuned mass damper in [3], is very accurate for realistic damping ratio values. It serves as a convenient starting point in designing an appropriate resonant filter, when a desired modal damping ratio ζ is prescribed.

5 Multi-Degree-of-Freedom Systems

Results for single-degree-of-freedom systems can often be calibrated so that they are representative for modal behavior in multi-degree-of-freedom systems. The typical problem is that the individual mode only accounts for part of the motion registered by a control sensor, and similarly the actuator force excites other modes as well, as illustrated in Fig. 1a. This is the so-called ‘spillover’ problem. A formulation of truncated modal analysis in which the motion represented by a small number of selected modes is supplemented by a quasi-static representation of the remaining modes goes back to the seventies, see e.g. [8]. In the following a quasi-static correction procedure is developed for resonant control.

5.1 Background Mode Correction in MDOF Systems

Let a multi-degree-of-freedom system be represented by the stiffness matrix $\mathbf{K}$ and the mass matrix $\mathbf{M}$. The displacements are denoted $\mathbf{q}$ and the corresponding external loads $\mathbf{F}$. In addition to the external load an actuator with control signal η is connected to the structure as described by the connectivity vector $\mathbf{w}$. The connectivity vector $\mathbf{w}$ contains the number 1 at the appropriate degree-of-freedom, if it is connected to the surroundings, and a set of numbers 1 and -1 for two degrees-of-freedom that are connected by an actuator. The corresponding frequency equation is

$$(\mathbf{K} - \omega^2 \mathbf{M}) \mathbf{q} = \mathbf{F} - \mathbf{w} G_{q\eta}(\omega) \eta. \quad (29)$$

The actuator is assumed to be controlled by a collocated signal, i.e. a signal generated by a sensor attached to the same degrees-of-freedom as the actuator. Thus, the actuator signal η is controlled by the frequency equation

$$G_{\eta\eta}(\omega) \eta = -G_{\eta q}(\omega) \mathbf{w}^T \mathbf{q}. \quad (30)$$

A modal representation of the response is introduced in the form

$$\mathbf{q} = \sum_j \mathbf{u}_j x_j, \quad \mathbf{u}_j^T \mathbf{M} \mathbf{u}_k = \delta_{jk}, \quad (31)$$

where $\mathbf{u}_j$ denotes the j 'th mode shape vector, and x_j the corresponding modal amplitude. The second equation, where δ_{jk} denotes Kronecker's delta, normalizes the modes corresponding to unit modal mass. The following analysis concentrates on control of the first mode, and an important parameter then is

$$v_1 = \mathbf{w}^T \mathbf{u}_1, \quad (32)$$

representing the displacement of the mode shape vector $\mathbf{u}_1$ across the actuator.

Control of the first mode is based on the corresponding modal equation, following from (29) by pre-multiplication with the mode shape vector $\mathbf{u}_1^T$,

$$G_{qq}(\omega) x_1 + v_1 G_{q\eta}(\omega) \eta = F_1, \quad (33)$$

where the modal frequency response is given in terms of the undamped modal frequency ω_1 as

$$G_{qq}(\omega) = \omega_1^2 - \omega^2 \quad (34)$$

and $F_1 = \mathbf{u}_1^T \mathbf{F}$ is the corresponding external modal load.

In the control equation (30) the displacement across the actuator $\mathbf{w}^T \mathbf{q}$ contains contributions from background modes as well as the mode to be controlled. For flexible structures it is important to include the effect of these ‘background modes’. A representation of this effect is obtained by constructing an approximate form of the displacement vector $\mathbf{q}$ from inversion of the homogeneous equation of motion (29),

$$\mathbf{w}^T \mathbf{q} = -\mathbf{w}^T (\mathbf{K} - \omega^2 \mathbf{M})^{-1} \mathbf{w} G_{q\eta}(\omega) \eta. \quad (35)$$

The inverse of the dynamic stiffness matrix can be expanded in terms of the mode shape vectors, whereby

$$(\mathbf{K} - \omega^2 \mathbf{M})^{-1} = \sum_j \frac{\mathbf{u}_j \mathbf{u}_j^T}{\omega_j^2 - \omega^2} \simeq \frac{\mathbf{u}_1 \mathbf{u}_1^T}{\omega_1^2 - \omega^2} + \sum_{j=2} \frac{\mathbf{u}_j \mathbf{u}_j^T}{\omega_j^2}. \quad (36)$$

In the second, approximate form, the contributions from the higher modes have been replaced by their static counterparts. When using the full expansion at zero frequency,

$$\mathbf{K}^{-1} = \sum_j \frac{\mathbf{u}_j \mathbf{u}_j^T}{\omega_j^2}, \quad (37)$$

the summation in (36) can be eliminated, whereby

$$(\mathbf{K} - \omega^2 \mathbf{M})^{-1} \simeq \frac{\mathbf{u}_1 \mathbf{u}_1^T}{\omega_1^2 - \omega^2} + \left(\mathbf{K}^{-1} - \frac{\mathbf{u}_1 \mathbf{u}_1^T}{\omega_1^2} \right). \quad (38)$$

In this form the term in the parentheses represents the effect of the higher modes.

When this representation is substituted into the expression (35) for $\mathbf{w}^T \mathbf{q}$, the following form is obtained

$$\mathbf{w}^T \mathbf{q} \simeq - \left[\frac{v_1^2}{G_{qq}(\omega)} + \kappa_1 \right] G_{q\eta}(\omega) \eta = v_1 x_1 - \kappa_1 G_{q\eta}(\omega) \eta, \quad (39)$$

where the denominator $G_{qq}(\omega)$ is the frequency function (34) of the first mode, and the modal response x_1 has been introduced from the homogeneous form of (33).

The parameter κ_1 , representing the effect of the higher modes, is defined as

$$\kappa_1 = \mathbf{w}^T \left(\mathbf{K}^{-1} - \frac{\mathbf{u}_1 \mathbf{u}_1^T}{\omega_1^2} \right) \mathbf{w} = \mathbf{w}^T \mathbf{K}^{-1} \mathbf{w} - (v_1/\omega_1)^2. \quad (40)$$

The parameter κ_1 has a direct physical interpretation. The first term is the displacement between the degrees of freedom of the sensor for a unit force exerted by the actuator, and the second term subtracts the part associated with the first mode. It is convenient to represent this parameter in the normalized form

$$\kappa = (\omega_1/v_1)^2 \kappa_1 = (\omega_1/v_1)^2 \mathbf{w}^T \mathbf{K}^{-1} \mathbf{w} - 1, \quad (41)$$

giving the relative extra flexibility contained in the higher modes.

The final form of the control equation for the first mode is now obtained by substituting the approximate displacement representation (39) into the control equation (30). The combined set of equations for the first mode and the controller then takes the form

$$\begin{aligned} G_{qq}(\omega) x_1 + v_1 G_{q\eta}(\omega) \eta &= F_1 \\ v_1 G_{\eta q}(\omega) x_1 + [G_{\eta\eta}(\omega) - \kappa_1 G_{\eta q}(\omega) G_{q\eta}(\omega)] \eta &= 0. \end{aligned} \quad (42)$$

It is seen that the resonator frequency function $G_{\eta\eta}(\omega)$ is modified by the term $\kappa_1 G_{\eta q}(\omega) G_{q\eta}(\omega)$, and the coupling is modified by the coefficient v_1 , representing the modal displacement across the actuator as defined in (32).

5.2 MDOF System with Acceleration Feedback

The calibration procedure consists in establishing an equivalence with the optimal control of the SDOF system presented in Sect. 4. The modal frequency function and the resonant controller are represented in the form

$$G_{qq}(\omega) = \omega_1^2 - \omega^2, \quad G_{\eta\eta}(\omega) = \omega_\eta^2 - \omega^2 + 2i\zeta_\eta\omega_\eta\omega. \quad (43)$$

In the case of acceleration feedback the sensor and actuator functions are given as

$$G_{\eta q}(\omega) = \omega^2, \quad v_1^2 G_{q\eta}(\omega) = \alpha_\eta \omega_\eta^2 + \beta_\eta 2i\zeta_\eta\omega_\eta\omega. \quad (44)$$

The calibration of optimal parameters consists of finding expressions for the controller frequency ω_η and damping ratio ζ_η as well as the gain parameters α_η and β_η in terms of a common gain parameter.

The first step is the identification of the equivalent frequency and damping parameters of the controller, represented by the terms in the square brackets in (42),

$$\begin{aligned} G_{\eta\eta}(\omega) - \kappa_1 G_{\eta q}(\omega) G_{q\eta}(\omega) \\ = \omega_\eta^2 - [1 + \kappa\alpha_\eta(\omega_\eta/\omega_1)^2]\omega^2 + 2i\zeta_\eta\omega_\eta\omega[1 - \kappa\beta_\eta(\omega/\omega_1)^2]. \end{aligned} \quad (45)$$

The last term in this expression is cubic in ω , and does not fit into the format established for the SDOF system. This term is mainly important around resonance, and thus the factor $(\omega/\omega_1)^2$ is set equal to 1 in the last square brackets. In the original SDOF format the frequency function of the controller was introduced in normalized form, and therefore the present modified frequency format is rewritten in normalized form with unit coefficient to ω^2 ,

$$\begin{aligned} G_{\eta\eta}(\omega) - \kappa_1 G_{\eta q}(\omega) G_{q\eta}(\omega) &= [1 + \alpha_\eta\kappa(\omega_\eta/\omega_1)^2] \\ &\times \left[\frac{\omega_\eta^2}{1 + \kappa\alpha_\eta(\omega_\eta/\omega_1)^2} - \omega^2 + 2i\zeta_\eta\omega_\eta\omega \frac{1 - \kappa\beta_\eta}{1 + \kappa\alpha_\eta(\omega_\eta/\omega_1)^2} \right]. \end{aligned} \quad (46)$$

The terms in the square brackets are now identified with the frequency function $G_{yy}(\omega)$ given in (6). This gives the two relations

$$\omega_c^2 = \frac{\omega_\eta^2}{1 + \kappa\alpha_\eta(\omega_\eta/\omega_1)^2}, \quad \zeta_c\omega_c = \zeta_\eta\omega_\eta \frac{1 - \kappa\beta_\eta}{1 + \kappa\alpha_\eta(\omega_\eta/\omega_1)^2}. \quad (47)$$

Similarly the product $G_{xy}(\omega)G_{yx}$ from the SDOF formulation (11) with $\gamma = 0$ can be identified with the off-diagonal product of the modified formulation, taking the form

$$\frac{\nu_1^2 G_{\eta q}(\omega) G_{q\eta}(\omega)}{1 + \kappa\alpha_\eta(\omega_\eta/\omega_1)^2} = \frac{\omega^2}{1 + \kappa\alpha_\eta(\omega_\eta/\omega_1)^2} [\alpha_\eta\omega_\eta^2 + \beta_\eta 2i\zeta_\eta\omega_\eta\omega]. \quad (48)$$

Comparison with $G_{xy}(\omega)G_{yx}(\omega)$ from (11) gives the two relations

$$\alpha\omega_c^2 = \frac{\alpha_\eta\omega_\eta^2}{1 + \kappa\alpha_\eta(\omega_\eta/\omega_1)^2}, \quad \beta\zeta_c\omega_c = \frac{\beta_\eta\zeta_\eta\omega_\eta}{1 + \kappa\alpha_\eta(\omega_\eta/\omega_1)^2}. \quad (49)$$

The four relations in (47) and (49) permit a parametric expression of the optimal MDOF gain parameters α_η , β_η and the controller frequency ω_η and the damping ratio ζ_η . It turns out to be convenient to formulate the parameter representation in terms of the equivalent SDOF gain parameter β .

The gain parameter α_η is determined by eliminating ω_c^2 between (47a) and (49a), whereby

$$\alpha_\eta = \alpha = \beta, \quad (50)$$

where the last relation follows from (23). The gain parameter β_η follows similarly by eliminating $\zeta_c \omega_c$ between (47b) and (49b),

$$\beta_\eta = \frac{\beta}{1 + \kappa\beta}. \quad (51)$$

These relations imply that $\alpha_\eta > \beta_\eta$, when the effect of increased flexibility from background modes is included. The classic tuned mass damper [1, 3] can be represented by the present acceleration feedback format with $\alpha = \beta$. It follows that this relation puts limitations on the optimality of the tuning, when applied to MDOF systems with influence from background modes.

The frequency ratio of the MDOF controller ω_η/ω_1 is determined from (49a), when written in the form

$$\left(\frac{\omega_\eta}{\omega_c}\right)^2 = \left(\frac{\omega_1}{\omega_c}\right)^2 \left(\frac{\omega_\eta}{\omega_1}\right)^2 = 1 + \kappa\beta \left(\frac{\omega_\eta}{\omega_1}\right)^2. \quad (52)$$

In this relation the frequency ratio $\omega_1/\omega_c = 1 + \beta$ corresponds to the optimal tuning frequency ratio (21) of the SDOF system. The equation then gives the optimal MDOF frequency as

$$\omega_\eta^2 = \frac{1}{1 - \frac{\kappa\beta}{(1 + \beta)^2}} \left(\frac{\omega_1}{1 + \beta}\right)^2. \quad (53)$$

The effect of the background modes is represented by the first factor. It vanishes for $\kappa = 0$ and leads to an increase in the tuning frequency ω_η with increasing influence of the background flexibility.

The damping ratio ζ_η is found from (49b) in the form

$$\zeta_\eta \omega_\eta = \frac{\beta}{\beta_\eta} \left[1 + \kappa\beta \left(\frac{\omega_\eta}{\omega_1}\right)^2 \right] \zeta_c \omega_c. \quad (54)$$

This relation can be rewritten by use of the β -ratio from (51) and the frequency ratio from (53). The result of the reduction is

$$\zeta_\eta \omega_\eta = \frac{1 + \kappa\beta}{1 - \frac{\kappa\beta}{(1 + \beta)^2}} \frac{\zeta_c \omega_1}{1 + \beta}. \quad (55)$$

The damping ratio ζ_c corresponds to optimal damping of a SDOF system and is given by (25). Also in this case the effect of the background modes is represented by the first factor.

6 Damping of Wind Excited Building

The efficiency and characteristics of the filtered acceleration feedback approach are illustrated in terms of the wind-excited benchmark building introduced in [9]. The height of the building is 306.1 m and it has 76 stories. The structure is modelled by 76 simple 2D beam elements, where the rotational degrees-of-freedom have been eliminated by static condensation. This leaves 76 degrees-of-freedom representing the horizontal displacement of each floor, see [9] for further details on the model. The time domain equations of motion for the structure with resonant control can be written in the compact form

$$\mathbf{M}_* \ddot{\mathbf{z}}(t) + \mathbf{C}_* \dot{\mathbf{z}}(t) + \mathbf{K}_* \mathbf{z}(t) = \mathbf{F}_*(t), \quad (56)$$

where the augmented vector $\mathbf{z} = [\mathbf{q}^T, \eta]^T$ contains the structural degrees-of-freedom and the control variable. The system matrices follow directly as

$$\begin{aligned} \mathbf{M}_* &= \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ -\mathbf{w}^T & 1 \end{bmatrix}, & \mathbf{C}_* &= \begin{bmatrix} \mathbf{C} & (\beta_\eta/v_1^2)2\zeta_\eta\omega_\eta\mathbf{w} \\ \mathbf{0}^T & 2\zeta_\eta\omega_\eta \end{bmatrix} \\ \mathbf{K}_* &= \begin{bmatrix} \mathbf{K} & (\alpha_\eta/v_1^2)\omega_\eta^2\mathbf{w} \\ \mathbf{0}^T & \omega_\eta^2 \end{bmatrix}, & \mathbf{F}_* &= \begin{bmatrix} \mathbf{F}_w \\ 0 \end{bmatrix}, \end{aligned} \quad (57)$$

where the external force vector $\mathbf{F}_w$ represents the wind loading and $\mathbf{0}$ is a 76×1 zero column vector. The actuator force, included in the last column of $\mathbf{C}_*$ and $\mathbf{K}_*$, follows from the last term in (29) which can be computed via the control variable as

$$F_c(t) = \frac{\alpha_\eta}{v_1^2} \omega_\eta^2 \eta(t) + \frac{\beta_\eta}{v_1^2} 2\zeta_\eta \omega_\eta \dot{\eta}(t). \quad (58)$$

Thus, the uncontrolled case is recovered for $\alpha_\eta = \beta_\eta = 0$.

6.1 Design of Resonant Control

The mode shapes $\mathbf{u}_j$ and natural frequencies ω_j of the undamped structure are shown in Fig. 6 for the first three modes. In the following the first mode is considered as the target mode for the resonant control. A force actuator is used to apply the desired resonant control. To demonstrate the influence of the background modes two locations of the force actuator are considered, as shown in Fig. 7. The largest relative displacement of mode 1 occurs between the top floors, and this actuator location is in the following referred to as case (a). The smallest relative displacement occurs between the two bottom floors, which is case (b). In both cases the actuator is placed in a Chevron brace so that the relative motion over the actuator is equal to the relative motion between the adjacent floors. Hereby, the connectivity vectors

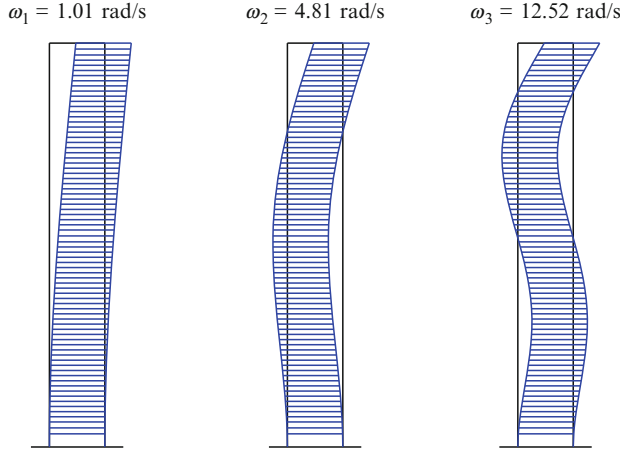


Fig. 6 Undamped mode shapes and natural frequencies for first three modes

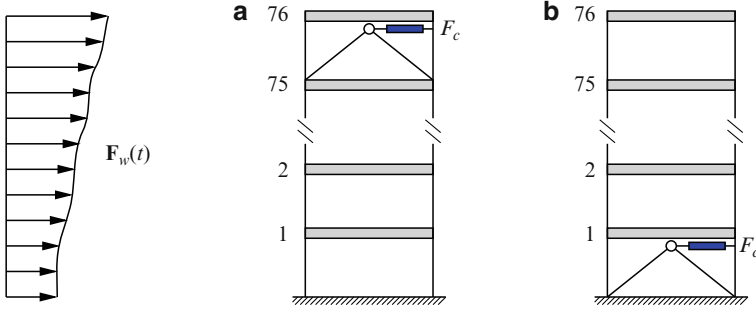


Fig. 7 76-story building with a force actuator in a Chevron brace acting between (a) top and (b) bottom floors

for case (a) and (b) are $\mathbf{w} = [0, \dots, -1, 1]^T$ and $\mathbf{w} = [1, 0, \dots, 0]^T$, respectively. For mode 1 the non-dimensional flexibility factor is $\kappa = 1.2125$ for case (a). For case (b) it is $\kappa = 10.371$ and thereby almost an order of magnitude larger than for case (a).

The sequential design procedure is described via the first eight columns of Table 2, where the desired modal damping ratio ζ_{des} of the target mode acts as the initial design variable. In the table four design situations with $\zeta_{des} = 0.04, 0.06, 0.08$ and 0.10 are presented. Because of the equal split of modal damping it follows from (28) that the damping parameter of the resonant control should be twice the desired damping ratio: $\zeta_c = 2\zeta_{des}$. Hereafter, the gain parameter β can be found by (25). The gain parameter can also be expressed directly in terms of the desired modal damping ratio as

$$\beta = \frac{8\zeta_{des}^2}{1 - 8\zeta_{des}^2}. \quad (59)$$

Table 2 Mode 1 damping ratios for 76-story building with resonant force actuator

ζ_{des}	ζ_c	β	ω_c/ω_1	β_η	α_η	ω_η/ω_1	ζ_η	$\zeta_1^\mp$	$\zeta_1^\mp _{\kappa=0}$
Actuator between top floors: $\kappa = 1.2125$									
0.04	0.08	0.0130	0.9872	0.0128	0.0130	0.9949	0.0819	0.0403 0.0401	0.0429 0.0357
0.06	0.12	0.0297	0.9712	0.0286	0.0297	0.9881	0.1265	0.0610 0.0606	0.0648 0.0507
0.08	0.16	0.0540	0.9488	0.0506	0.0540	0.9780	0.1757	0.0826 0.0819	0.0858 0.0643
0.10	0.20	0.0870	0.9200	0.0787	0.0870	0.9640	0.2317	0.1058 0.1046	0.1052 0.0771
Actuator between bottom floors: $\kappa = 10.371$									
0.04	0.08	0.0130	0.9872	0.0114	0.0130	1.0590	0.0974	0.0405 0.0402	0.0505 0.0164
0.06	0.12	0.0297	0.9712	0.0227	0.0297	1.1527	0.1862	0.0620 0.0612	0.0657 0.0170
0.08	0.16	0.0540	0.9488	0.0346	0.0540	1.3469	0.3543	0.0852 0.0843	0.0717 0.0164
0.10	0.20	0.0870	0.9200	0.0457	0.0870	1.8909	0.7818	0.1078 0.1086	0.0716 0.0153

The rest of the design parameters are expressed in terms of β . The frequency of the resonant filter in the single-degree-of-freedom case follows from Table 1 as $\omega_c = \omega_1/(1 + \beta)$, and finally the parameters α_η , β_η , ω_η and ζ_η for the multi-degree-of-freedom system are determined from (50), (51), (53) and (55), respectively.

6.2 Root Locus Analysis

The free vibration properties are illustrated in terms of a root locus analysis, where the complex valued natural frequencies ω are plotted in the complex plane for increasing gain β . The natural frequency is obtained by solving the corresponding eigenvalue problem that follows from the frequency domain representation of the equation of motion in (56) when both the external loading $\mathbf{F}$ and the structural damping $\mathbf{C}$ are omitted. Figure 8 shows the root loci for the two modes associated with the first vibration form of the structure. The solid curve represents the trajectories of the two natural frequencies of mode 1 when including the background flexibility, while the dashed curves represent the case without quasi-static correction. The markers represent the complex natural frequencies for the four cases in Table 2: $\zeta_{des} = 0.04, 0.06, 0.08$ and 0.1 .

Figure 8a shows the root locus diagram when the actuator is operating between the top floors and it is observed that the solid curves follow the 45° trajectories similar to the ideal case in Fig. 3b. The actual modal damping ratios are given in the last two columns in Table 2. It is seen that for $\kappa = 1.2125$ the two damping ratios

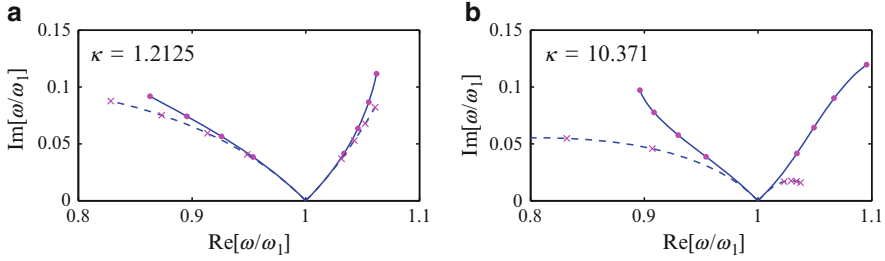


Fig. 8 Root locus diagram for mode 1. (a) Force actuator between top floors and (b) between bottom floors

are practically identical and equal to the desired value $\zeta_{des} = 0.04, 0.06, 0.08$ and 0.1 . In the case without quasi-static correction for higher modes ($\kappa = 0$) the optimal equal split property is not attained, resulting in a reduced damping efficiency of the control. This effect can also be seen for the dashed loci in Fig. 8a, where the right trajectory develops too little damping.

Figure 8b shows the root locus diagram when the actuator is attached between the bottom floors. The large flexibility parameter $\kappa = 10.371$ implies that the solid trajectories in the figure deviate slightly from the ideal curves for large damping ratios. However, as seen in Table 2 the two damping ratios are still equal, although slightly larger than the desired values. In this case the importance of the quasi-static correction becomes clear when comparing to $\kappa = 0$. In that case the left dashed locus in Fig. 8b is very large while the right is very small. As seen from the last column of Table 2 the balance between the two damping ratios is completely lost in this case. Thus, when the actuator is located where the relative modal deformation is small, the κ -value becomes relatively large and the associated correction for background modes is critically important.

6.2.1 Response Analysis

The root locus analysis describes the behavior of the modal damping properties, where robust design relies on equal modal damping, which defines the filter frequency ω_c as shown in Sect. 4. However, the remaining control parameters α and ζ_c are determined to secure optimal frequency response characteristics, as shown in Fig. 4. The efficiency of the resonant control strategy for harmonic excitation of multi-degree-of-freedom structures has been demonstrated in [4], and the present example therefore verifies the performance of the resonant control strategy for random loading, in this case represented by wind excitation. The time history of the wind load is presented in [9] and has been obtained by wind tunnel tests. The wind load on the top floor is shown in Fig. 9.

The time domain equation of motion (56) is solved by the standard (average acceleration) Newmark time integration procedure, see e.g. [10]. The time increment

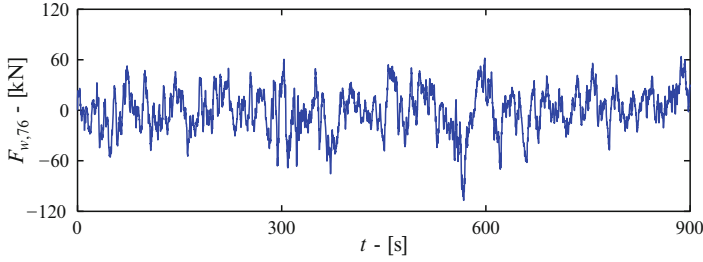


Fig. 9 Wind load history for top floor

$\Delta t = 0.1333$ s is dictated by the sampling frequency of the wind data and corresponds to approximately 47 time increments per mode 1 vibration period of the structure. Structural damping is contained in $\mathbf{C}$ and yields 1% of critical damping in the first five modes, see [9].

In [9] a total of 12 performance criteria are presented, describing the root mean square (rms) and the peak values of the response of both the structure and the control force. Only some of these criteria are used in the present example. The maximum rms values of the structural response are evaluated in terms of the top floor response

$$J_1 = \sigma_{\ddot{q},75}/\sigma_{\ddot{q},75}^0, \quad J_2 = \sigma_{q,76}/\sigma_{q,76}^0, \quad (60)$$

where $\sigma_{q,j}$ and $\sigma_{\ddot{q},j}$ are the rms displacement and acceleration of the j 'th floor, while superscript 0 refers to the case without control. Note that the acceleration criterion is based on floor 75, since floor 76 is the roof and therefore unoccupied. The third performance criterion describes the power consumption of the actuator and is determined by the time integral

$$J_3 = \sqrt{\frac{1}{T} \int_0^T \left(\dot{x}(t) F_c(t) \right)^2 dt}, \quad [J_3] = \text{kN m/s}, \quad (61)$$

where $\dot{x} = \mathbf{w}^T \dot{\mathbf{q}}$ is the velocity of the actuator and $T = 900$ s is the total length of the simulation. The integral is computed by the trapezoidal rule.

The peak response of the structure is described by

$$J_4 = \ddot{q}_{p,75}/\ddot{q}_{p,75}^0, \quad J_5 = q_{p,76}/q_{p,76}^0, \quad (62)$$

where $\ddot{q}_{p,j}$ and $q_{p,j}$ are the peak acceleration and displacement of the j 'th floor. The peak power consumption of the actuator is finally defined as

$$J_6 = \max \left| \dot{x}(t) F_c(t) \right|, \quad [J_6] = \text{kN m/s}, \quad (63)$$

where $|\cdot|$ denotes the absolute value.

Table 3 Performance criteria for resonant controller

ζ_{des}	J_1	J_2	J_3	J_4	J_5	J_6
Actuator between top floors: $\kappa = 1.2125$						
0.04	0.468	0.601	7.029	0.474	0.709	43.096
0.06	0.391	0.557	7.729	0.422	0.695	44.184
0.08	0.341	0.533	8.287	0.416	0.668	54.342
0.10	0.304	0.520	8.737	0.397	0.648	69.063
Actuator between bottom floors: $\kappa = 10.371$						
0.04	0.464	0.598	7.912	0.461	0.707	46.543
0.06	0.389	0.553	9.402	0.412	0.694	52.036
0.08	0.344	0.529	11.108	0.409	0.679	63.216
0.10	0.318	0.514	13.081	0.398	0.668	96.043
Sample controller [9]						
	0.369	0.578	11.99	0.381	0.717	71.96

The six performance criteria are summarized in Table 3 for the two actuator locations and the four desired damping ratios also used in the root locus analysis. The bottom row in the table gives the corresponding performance criteria for the LQG sample controller in the benchmark paper [9]. It is seen that the structural rms values J_1 and J_2 are reduced with increasing desired damping and good performance is observed compared to the LQG reference case in [9]. It should be noted that the reduction in structural response is practically the same for the two actuator locations. In general it is observed that the influence of κ is mainly on the power needed to realize the required actuator force, in the Table represented by J_3 and J_6 . It is also noted that the reduction in the peak values is not as significant as for the rms values. This is because the resonant controller is designed with respect to stationary response, whereby the ability to reduce peak response from transients, such as wind gusts and earthquakes, is limited.

7 Conclusions

A design principle for resonant control of a SDOF system by second order filters has been presented. It covers the two cases of modified acceleration feedback and modified displacement feedback and includes the classic tuned mass damper as a special case of acceleration feedback. A set of explicit formulae is obtained for the optimal control parameters by combining the root locus property of equal modal damping with optimal response curve characteristics.

The procedure is extended to collocated resonant control of the lower modes of MDOF systems. In this case the response recorded by the sensor typically includes contributions from the motion of higher modes. An explicit design procedure for the control parameters of MDOF systems has been developed that includes the effect of the higher modes via a quasi-static approximation. A numerical example

in the form of a benchmark problem involving turbulent wind load on a high-rise building illustrates the potential of resonant control, and demonstrates the importance of including a quasi-static correction of the control parameters, in particular when the control force is applied close to the supports of the structure.

References

1. Den Hartog, J.P.: *Mechanical Vibrations*, 4th edn. McGraw-Hill, New York (1956)
2. Preumont, A.: *Vibration Control of Active Structures. An Introduction*, 2nd edn. Kluwer, Dordrecht (2002)
3. Krenk, S.: Frequency analysis of the tuned mass damper *J. Appl. Mech. ASME* **72**, 936–942 (2005)
4. Krenk, S., Høgsberg, J.: Optimal resonant control of flexible structures *J. Sound Vib.* **323**, 530–554 (2009)
5. Ozer, M.B., Royston, T.J.: Application of Sherman-Morrison matrix inversion formula to damped vibration absorbers attached to multi-degree of freedom systems. *J. Sound Vib.* **283**, 1235–1249 (2005)
6. Ozer, M.B., Royston, T.J.: Extending Den Hartog's vibration absorber technique to multi-degree of freedom systems. *J. Vib. Acoust.* **127**, 341–350 (2005)
7. Krenk, S., Høgsberg, J.: Tuned mass absorbers on damped structures under random load. *Probabilistic Eng. Mech.* **23**, 408–415 (2008)
8. Hansteen, O.E., Bell, K.: Accuracy of mode superposition analysis in structural dynamics. *Earthquake Eng. Struct. Dyn.* **7**, 405–411 (1979)
9. Yang, J.N., Agrawal, A.K., Samali, B., Wu, J.-C.: Benchmark problem for response control of wind-excited tall buildings. *J. Eng. Mech.* **130**, 437–446 (2008)
10. Geradin, M., Rixen, D.: *Mechanical Vibrations*, 2nd edn. Wiley, Chichester (1997)

Importance Sampling of Nonlinear Structures Using Adapted Process

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Abstract The reliability analysis of nonlinear-hysteretic structures subjected to uncertain earthquake excitations is an important but challenging problem in stochastic structural dynamics and performance-based earthquake engineering. The ‘first passage probability’, i.e., the probability that some response quantities of interest (e.g., interstory drifts) exceed specified threshold levels, is often of concern. Direct Monte Carlo simulation is a versatile method for estimation, but it is not efficient when the failure probability is small, which is often encountered in engineering applications. Importance sampling is a popular variance reduction technique where a change of distribution is applied in order to achieve a higher failure rate in the samples and hence variance reduction. Its success hinges on a proper choice of a user-defined ‘importance sampling distribution’. A popular choice makes use of ‘design point excitations’ that are local most probable points within the failure region in the stochastic load space. Design point excitations participate in the excitation to create large response, leading to a larger failure rate in the samples. Their use has led to tremendous variance reduction for linear systems. For nonlinear-hysteretic systems, however, recent research shows that the gain in efficiency is much less than their linear counterpart, essentially because plastic excursions cause random phase shifts in the response that de-synchronize it from the design point excitations, undermining their effectiveness in creating large response. This work presents a new approach for constructing the importance sampling density for estimating the first passage probability of nonlinear-hysteretic systems subjected to stochastic earthquake excitations. Instead of using fixed design point excitations, the importance sampling distribution is constructed using an ‘adapted process’ whose future action can be updated based on information up to the present. Choosing the adapted process involves designing an adaptive control force algorithm in a stochastic environment that targets to drive the response to first passage failure based on information up to the present. Theoretical and computational issues related to the

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proposed importance sampling method shall be investigated. A stochastic control algorithm for generating the adapted process is presented and its variance reduction capability is appraised.

1 Introduction

Determining the first passage probability of nonlinear hysteretic structures remains a challenging computational problem in stochastic dynamics [1–5]. The complexity involved is featured by a large number of random variables (theoretically infinite), nonlinear/hysteretic behavior and a large number of degrees of freedom. The first renders geometric intuitions in low dimensions less useful or sometimes misleading in high dimensions [6, 7]. Nonlinearity, hysteresis and a large number of degrees of freedom make the understanding of system behavior more involved and in many cases analytical solution almost impossible.

Importance sampling method is an established variance reduction technique for estimating the expectation of functions of random variables [8, 9]. Its success hinges on the choice of importance sampling distribution (ISD). In structural reliability research, many schemes for constructing the ISD have appeared in the literature. A popular choice is a weighted sum of Gaussian distributions centered among design points [10, 11] or random pre-samples [12–14]. Importance sampling has been successfully applied to static reliability problems but the same is not true for dynamic reliability problems that often involve a large number of random variables in the stochastic description of the excitation.

A recent benchmark study by Schueller and co-workers [15] indicated that advanced Monte Carlo methods have shown promise for tackling complex systems. Subset Simulation, line sampling, and collectively methods that make use of Markov Chain Monte Carlo have demonstrated their variance reduction capability while retaining certain robustness in applicability. These methods explore the progressive failure nature of reliability problems and do not significantly rely on knowing system behavior. The benchmark study also revealed a big gap between the extent of variance reduction that can currently be achieved for linear systems versus nonlinear hysteretic systems. For example, Problem 3 in the benchmark study indicates that, using Subset Simulation [16] or line sampling [17] that are based on Markov Chain Monte Carlo, one can achieve a unit c.o.v. (standard error = unit c.o.v./square root of no. of samples) of 30–40 for a target failure probability of the order of 10^{-4} . This translates into about 5,000 samples to achieve a 50% c.o.v. in the failure probability estimate. In contrast, using methods such as importance sampling or line sampling that incorporate information about system behavior, one can achieve for linear systems a unit c.o.v. of 2–5 regardless of failure probability level, i.e., at most 100 samples required to achieve 50% c.o.v. in the estimate. A large variance reduction has been achieved for linear systems but there is much room to improve for general nonlinear systems. Notably, linear system behavior is analytically tractable

and very efficient importance sampling methods have been developed [18–20]. For nonlinear-elastic systems, bounds for the maximum response have been obtained [21]; the design points have been recently obtained and applied to random vibration analysis [22]. The analysis of nonlinear-hysteretic systems is most difficult, although hysteretic structures are frequently encountered in applications. The design points are not known analytically and the geometry of the failure region is far more complex than that of an elastic system. Westermo [23] considered maximizing the input power within the class of excitations spanned by the displacement and velocity response. Approximate periodic solutions were obtained numerically for the critical response of single-degree-of-freedom (SDOF) elasto-plastic systems. Equivalent linearization [24] has been applied for approximate solution in the frequency domain [25, 26].

The trade-off between efficiency and robustness of a simulation method justifies methods to be more application-focused to improve efficiency at the expense of generality in application. An effort was initiated to develop an importance sampling method for nonlinear hysteretic structures. The ultimate aim is to achieve a level of variance reduction similar to that for linear systems, or otherwise find out what methods do not work and the reasons behind. The first attempt was to determine the design points of single-degree-of-freedom (SDOF) elasto-plastic structures and use them to construct an importance sampling density, speculating significant variance reduction similar to the linear case. It turned out that efficient solutions for design points are possible in this case [27, 28]. Using the design points for importance sampling, however, led to only limited variance reduction compared to the linear case [29]. Further investigations showed that the actions of the design points were rendered ineffective by random phase shifts associated with plastic excursions as well as opposing plastic excursions that cancelled out each other. This suggested naturally a design point that could ‘adapt’ and synchronize its actions with the response.

Although a ‘stochastic design point’ sounds subtle at first glance because the design point plays the role of the mean shift of the importance sampling distribution, detailed arguments support its legitimacy when its components bear a causal property. Such a process is in fact called an adapted process, and its concept is long-rooted in the celebrated Girsanov Theorem [30]. The use of adapted process opens up new variance reduction possibilities for hysteretic structures and, in general, complex causal systems. A stochastic control algorithm is presented in this work for the adapted process, which appears a natural choice because designing the adapted process is equivalent to designing a control law to achieve certain purposes with minimum energy under a stochastic environment where the future response is influenced not only by the control force but also by random white noise disturbances. Although stochastic control is an established area, ‘off-the-shelf’ control laws are not available for first passage problems. An optimal control law (in some heuristic sense) is developed and its variance reduction efficiency shall be investigated.

2 Importance Sampling Method

Consider a SDOF elasto-plastic structure subjected to white noise excitations. The displacement response of the elasto-plastic structure follows the governing equation

$$\ddot{x}(t) + 2\zeta\omega \dot{x}(t) + F_r = W(t), \quad x(0) = \dot{x}(0) = 0 \quad (1)$$

where ω and ζ are the natural frequency and critical damping ratio of the associated linear system at low amplitudes; F_r is the restoring force that follows an elasto-plastic hysteretic rule: $dF_r = \omega^2 dx$ if $|F_r| < \omega^2 b_0$ and $dF_r = 0$ otherwise; b_0 is the first yield displacement. The white noise excitation is modeled digitally in the time domain by

$$W(t_i) = \sqrt{\frac{2\pi S}{\Delta t}} Z_i \quad (2)$$

where Δt is the sampling time; S is the spectral intensity; $\{Z_i : i = 1, \dots, n\}$ are independent and identically distributed (i.i.d.) standard Gaussian random variables. Failure is defined as the first passage of the displacement response over the double barrier $\pm b_F$ within a given duration of interest $n\Delta t$:

$$P_F = P \left(\bigcup_{i=1}^n |x(t_i)| > b_F \right) = E [I(\mathbf{Z} \in F)] \quad (3)$$

The last expression in (3) views the failure probability as an expectation of the indicator function $I(\mathbf{Z} \in F)$ where $F = \{\mathbf{z} \in R^n : \bigcup_{i=1}^n |x(t_i; \mathbf{z})| > b_F\}$ denotes the failure region; $x(t_i; \mathbf{z})$ denotes explicitly the response at the i -th time step to the realization of white noise corresponding to $\mathbf{z} = [z_1, \dots, z_n]$. The expectation is taken with $\mathbf{Z} = [Z_1, \dots, Z_n]$ distributed as a standard Gaussian vector, i.e., with probability density function (PDF)

$$\phi_n(\mathbf{z}) = (2\pi)^{-n/2} \exp \left(-\frac{1}{2} \sum_{i=1}^n z_i^2 \right) \quad (4)$$

Importance sampling [8, 9] is a popular technique for variance reduction in reliability problems, especially for rare failure events where the computational effort required by direct Monte Carlo is prohibitive. The idea is to generate samples from a properly chosen distribution that leads more frequently to failure. Let $f(\mathbf{z})$, called the importance sampling density (ISD), be a PDF whose support covers the failure region. By a change of probability distribution from ϕ_n to f , the failure probability is expressed as

$$P_F = E[I(\mathbf{Z} \in F)] = \int I(\mathbf{z} \in F) \frac{\phi_n(\mathbf{z})}{f(\mathbf{z})} f(\mathbf{z}) d\mathbf{z} = E_f \left[I(\mathbf{Z}' \in F) \frac{\phi_n(\mathbf{Z}')}{f(\mathbf{Z}')} \right] \quad (5)$$

where in the last expectation $\mathbf{Z}'$ is distributed as f instead of ϕ_n . The failure probability is then estimated via statistical averaging by

$$\tilde{P}_F = \frac{1}{N} \sum_{k=1}^N I(\mathbf{Z}_k' \in F) \frac{\phi_n(\mathbf{Z}_k')}{f(\mathbf{Z}_k')} \quad (6)$$

with $\{\mathbf{Z}_k': k = 1, \dots, N\}$ being i.i.d. (independent and identically distributed) samples generated according to f . For a proper choice of f , $\tilde{P}_F$ is an unbiased estimator for P_F . Its statistical error is often measured by its coefficient of variation (c.o.v.), δ , defined as the ratio of its standard deviation to its mean. Due to the use of i.i.d. samples, δ is inversely proportional to $\sqrt{N}$ and can be expressed as

$$\delta = \frac{\Delta}{\sqrt{N}} \quad (7)$$

where Δ is the c.o.v. of $I(\mathbf{Z}' \in F) \phi_n(\mathbf{Z}')/f(\mathbf{Z}')$, referred as the ‘unit c.o.v.’ of the importance sampling estimator. The unit c.o.v. Δ is sample-size independent and is often used for measuring the efficiency of the sampling method [15, 31]. It can be readily shown that the following identity holds, which provides insights for variance reduction [7]:

$$\Delta^2 + 1 = \frac{\Delta_{R|F}^2 + 1}{Q_F} \quad (8)$$

where $\Delta_{R|F}$ is the conditional c.o.v. of the ‘importance sampling quotient’ $R(\mathbf{Z}') = \phi_n(\mathbf{Z}')/f(\mathbf{Z}')$ given that $\mathbf{Z}' \in F$; Q_F is the probability that $\mathbf{Z}' \in F$, referred here as the ‘failure rate’ (to distinguish it from the failure probability P_F). For direct Monte Carlo, $Q_F = P_F$ (no improvement in failure rate) and $\Delta_{R|F} = 0$ (since $R = \phi_n/f = \phi_n/\phi_n \equiv 1$) and so $\Delta^2 = 1/P_F - 1 = (1 - P_F)/P_F$ which checks with the classical expression for unit c.o.v. of direct Monte Carlo estimator. The identity in (8) suggests two general objectives in the choice of ISD: (1) to increase Q_F by producing more samples in the failure region and (2) to reduce $\Delta_{R|F}$ by a proper choice of the functional form of f . The former has been the major consideration in the structural reliability literature, although the latter has important significance for stochastic simulation problems that are featured by a large number n (theoretically infinite) of random variables [6, 7]. Most often, the ISD is chosen as a Gaussian PDF or a weighted sum of Gaussian PDFs with unit covariance matrix; the choice of ISD then reduces to the choice of the mean vector(s) of the Gaussian PDF(s).

2.1 Distribution Shifted to a Fixed Point

Consider first the classical case when the ISD is chosen as a Gaussian PDF centered at some fixed point $\hat{\mathbf{z}} = [\hat{z}_1, \dots, \hat{z}_n]$, i.e.,

$$f(\mathbf{z}) = \phi_n(\mathbf{z} - \hat{\mathbf{z}}) = (2\pi)^{-n/2} \exp \left[-\frac{1}{2} \sum_{i=1}^n (z_i - \hat{z}_i)^2 \right] \quad (9)$$

The failure probability is then expressed as

$$\begin{aligned} P_F &= E_f \left[I(\mathbf{Z}' \in F) \frac{\phi_n(\mathbf{Z}')}{f(\mathbf{Z}')} \right] = E \left[I(\hat{\mathbf{z}} + \mathbf{Z} \in F) \frac{\phi_n(\hat{\mathbf{z}} + \mathbf{Z})}{\phi_n(\mathbf{Z})} \right] \\ &= E \left[I(\hat{\mathbf{z}} + \mathbf{Z} \in F) \exp \left(-\frac{1}{2} \sum_{i=1}^n \hat{z}_i^2 - \sum_{i=1}^n \hat{z}_i Z_i \right) \right] \end{aligned} \quad (10)$$

where under $E[\cdot]$ the vector $\mathbf{Z}$ is distributed as a n -dimensional standard Gaussian vector with independent components. The failure probability is then estimated by averaging the term inside the expectation over i.i.d. samples of $\mathbf{Z}$. Note that during importance sampling the structure is subjected to the forcing corresponding to $\mathbf{Z}' = \hat{\mathbf{z}} + \mathbf{Z}$, which represents the combined action of the design point excitation and the white noise ($\mathbf{Z}$).

2.2 Distribution Shifted to an Adapted Process

Consider now allowing $\hat{\mathbf{z}} = [\hat{z}_1, \dots, \hat{z}_n]$ to depend on $\mathbf{Z} = [Z_1, \dots, Z_n]$ and viewing $\{\hat{z}_i: i = 1, 2, \dots\}$ as a discrete-time stochastic process. Specifically, $\hat{z}_1$ is fixed, and for every $i = 1, 2, \dots$, $\hat{z}_{i+1}$ depends only on $\{Z_1, \dots, Z_i\}$ but not on $\{Z_{i+1}, Z_{i+2}, \dots\}$. That is, the next future value depends on the information only up to the present. In the theory of stochastic process $\hat{\mathbf{z}} = [\hat{z}_1, \dots, \hat{z}_n]$ is called an adapted (or predictable) process. In our context, $\hat{\mathbf{z}}$ may be viewed as an adaptive control force in a stochastic environment. In contrast to a design point that is a fixed vector, the choice of $\hat{\mathbf{z}}$ as an adapted process involves the design of a set of rules (control law) defining how a realization of the process is generated.

For a given adapted process $\hat{\mathbf{z}}$, consider a transformation from $\mathbf{Z}$ to $\mathbf{Z}'$ defined by

$$\mathbf{Z}' = \hat{\mathbf{z}}(\mathbf{Z}') + \mathbf{Z} \quad (11)$$

where the sequence $\{Z_1', Z_2', \dots\}$ is generated sequentially from $\{Z_1, Z_2, \dots\}$ as follows:

$$\begin{aligned} Z_1' &= Z_1 + \hat{z}_1; Z_2' = Z_2 + \hat{z}_2(Z_1'); Z_3' = Z_3 + \hat{z}_3(Z_1', Z_2'); \\ Z_4' &= Z_4 + \hat{z}_4(Z_1', Z_2', Z_3'); \dots \end{aligned} \quad (12)$$

In general, for $i \geq 1$,

$$Z_{i+1}' = Z_{i+1} + \hat{z}_{i+1}(Z_1', \dots, Z_i') = Z_{i+1} + \hat{z}_{i+1}(\mathbf{Z}'^{(i)}) \quad (13)$$

where $\mathbf{Z}'^{(i)} = [Z_1', \dots, Z_i']$ denotes the components of $\mathbf{Z}'$ up to the i -th time step; a similar notation applies for other processes.

To obtain the PDF for $\mathbf{Z}'$, note that during importance sampling, $\{Z_1, \dots, Z_n\}$ are i.i.d. standard Gaussian. Given $\{Z_1', \dots, Z_i'\}$, $\hat{z}_{i+1}(\mathbf{Z}'^{(i)})$ is already determined, and hence Z_{i+1}' in (13) is Gaussian with mean $\hat{z}_{i+1}(\mathbf{Z}'^{(i)})$ and unit variance. The conditional PDF of Z_{i+1}' given $\mathbf{Z}'^{(i)} = [z_1', \dots, z_i']$ is thus given by

$$p(z_{i+1}' | \mathbf{z}'^{(i)}) = \phi_1(z_{i+1}' - \hat{z}_{i+1}(\mathbf{z}'^{(i)})) \quad (14)$$

where $\phi_1(\cdot)$ denotes the one-dimensional standard Gaussian PDF. Here, the lower-case letters denote the input argument of the PDF for the random vectors denoted by upper-case letters. We also use $p(\cdot)$ to denote a generic probability density function, and $p(\cdot|\cdot)$ for the conditional counterpart. The joint PDF for the random vector $\mathbf{Z}'$ can be obtained by sequentially conditioning on $\mathbf{Z}'^{(i)}$ ($i = n-1, n-2, \dots$):

$$p(\mathbf{z}') = p(\mathbf{z}'^{(n)}) = p(z_n' | \mathbf{z}'^{(n-1)}) p(\mathbf{z}'^{(n-1)}) = \dots = p(z_1') \prod_{i=2}^n p(z_i' | \mathbf{z}'^{(i-1)}) \quad (15)$$

Using (14), this becomes (abbreviating $\hat{z}_i(\mathbf{z}'^{(i-1)})$ as $\hat{z}_i$)

$$p(\mathbf{z}') = \phi_1(z_1' - \hat{z}_1) \prod_{i=2}^n \phi_1(z_i' - \hat{z}_i) = \phi_n(\mathbf{z}' - \hat{\mathbf{z}}) = (2\pi)^{-n/2} \exp \left[-\frac{1}{2} \sum_{i=1}^n (z_i' - \hat{z}_i)^2 \right] \quad (16)$$

which is the same expression as if $\hat{\mathbf{z}}$ were fixed.

In summary, it is legitimate to make a change of distribution to a Gaussian PDF centered at a stochastic vector $\hat{\mathbf{z}}$ when it has a causal structure; the corresponding PDF takes the same form as the one with a shift of a deterministic mean vector. The resulting ISD is no longer Gaussian in $\mathbf{z}'$, however, because $\hat{\mathbf{z}}$ now depends on $\mathbf{z}'$. Provided that $\|\hat{\mathbf{z}}\| < \infty$ almost surely, the importance sampling estimator is valid with a finite variance. The continuous-time or infinite dimensional ($n \rightarrow \infty$) analogue of this result is the Girsanov Theorem [30, 32].

3 Stochastic Control Approach

The adapted process acts as a feed-forward control force in the time domain. Its design involves the specification of a stochastic process, or a control law, in contrast to the choice of design point excitation that only involves specification of a fixed vector. Two issues need to be resolved: (1) to specify a practical design goal and (2) to design an algorithm that meets the specified goal. Ideally, the design goal is to

minimize the unit c.o.v. Δ of the failure probability estimate. This is not practical, however, because Δ is analytically intractable; computationally the estimation of Δ is even more difficult than the failure probability. A practical goal is one that can be readily analyzed so that specific algorithms can be devised. According to (8), Δ can be reduced by increasing Q_F and reducing $\Delta_{R|F}$. As Q_F and $\Delta_{R|F}$ are difficult to analyze and control directly, we shall work with their proxies instead. We take the energy of adapted process as a proxy for $\Delta_{R|F}$, which can be reasoned from the variance of the exponential term in (10). The design goal is to grow the response to excursion with as less energy as possible. These two objectives are often conflicting. For a given class of control strategy one does not know a priori the optimal combination of Q_F and $\Delta_{R|F}$ that minimizes Δ . A prudent choice should trade-off between Q_F and $\Delta_{R|F}$ in order to reduce Δ .

An attempt was made in [33] to design the adapted process using heuristic rules, which was theoretically simple because it only involved a set of logic rules and the algorithm parameters could be assigned in an intuitive manner. The decision rule was based on the adapted process acting alone and ignored the effect of the white noise excitation that actually acts together with the adapted process. It is thus of interest to explore possible gain in efficiency when the decision rule takes effect of the white noise into account. A stochastic control approach provides a rigorous machinery for this, which turns out to be more generic and allows for further development.

We first focus on the up-crossing problem and then extend the results to apply for a double-barrier problem. We consider the stochastic control problem of driving the response to have large positive plastic excursion, assuming the current state is in the linear regime. The objective is to drive the response from the current state in the linear regime to yielding. This objective does not directly address the original objectives in the first passage problem but it simplifies the design of controller. For the purpose of deriving the control law we shall work in continuous-time as the derivation is more elegant. Under importance sampling the governing equation in the linear regime is given by

$$\ddot{x}(t) + 2\zeta\omega\dot{x}(t) + \omega^2x(t) = W(t) + u(t) \quad (17)$$

where $x(t)$ is measured relative to the current neutral axis; $u(t)$ is the adapted process in the time domain, related to that in the standard Gaussian space by $u(t_k) = \sqrt{2\pi S/\Delta t} \hat{z}_k$. Interpreting in the Ito sense, $\dot{x}$ follows the stochastic differential equation

$$d\dot{x} = [u - \mu(x, \dot{x})]dt + \sigma dB \quad (18)$$

where $\mu(x, \dot{x}) = \omega^2x + 2\zeta\omega\dot{x}$, $\sigma = \sqrt{2\pi S}$ and $dB = Wdt$ is the standard Brownian motion increment ($E[(dB)^2] = dt$).

Consider designing a control force that depends on the current state ($x(t)$, $\dot{x}(t)$). For a given future realization define the objective function as

$$J(u) = \frac{1}{2} \int_t^\tau u(s)^2 ds + \lambda_1 \tau - \lambda_2 [\dot{x}(\tau)]_+ \quad (19)$$

where $\lambda_1, \lambda_2 > 0$ are positive constants to be specified; $\tau = \inf \{s > t: |x(s)| > b_0\}$ is the (random) time at which the response exits the linear-elastic regime, assuming $(x(t), \dot{x}(t))$ is in the linear-elastic regime; $[\cdot]_+ = \max(\cdot, 0)$ gives the value of its argument when positive and zero otherwise. The objective function $J(u)$ increases with the energy of the adapted process u until yielding. It also increases with the time until yielding, through the second term in (19). The third term decreases with the exit velocity at positive yielding (with positive plastic displacement) and is zero if yielding occurs with negative exit velocity (and hence causing negative plastic displacement). The third term thus favors positive yielding and remains neutral to negative yielding.

Since $J(u)$ depends on the stochastic future the actual objective function for design shall be expressed through its expectation conditional on the current state:

$$E[J(u)|x(t), \dot{x}(t)] = E \left[\frac{1}{2} \int_t^\tau u(s)^2 ds + \lambda_1 \tau - \lambda_2 [\dot{x}(\tau)]_+ \middle| x(t), \dot{x}(t) \right] \quad (20)$$

Minimizing $E[J(u)|x(t), \dot{x}(t)]$ will therefore achieve three objectives through the three terms discussed, respectively: (1) reducing the control effort to yielding; (2) reducing the time to yielding; and (3) promoting positive yielding. Relative importance of these three objectives are controlled by a proper choice of the parameters λ_1 and λ_2 . We shall come back to this after we have derived the control law.

3.1 Bellman's Equation

The minimization should be performed on the space of all admissible control strategies that depend only on the information up to the present. The stochastic control problem corresponds to one of an indefinite time horizon, i.e., the time span over which the performance is evaluated is not fixed but is dependent on the stopping time τ . With a time invariant system it can be argued that the control law and the minimum of the objective function depend only on the current state but not explicitly on time t [34]. Let $V(x, \dot{x})$ be the minimum, i.e.,

$$V(x, \dot{x}) = \min_u E[J(u)|x, \dot{x}] \quad (21)$$

Assuming V is sufficiently smooth to apply Ito Calculus,

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial \dot{x}} d\dot{x} + \frac{1}{2} \frac{\partial^2 V}{\partial \dot{x}^2} \sigma^2 dt \quad (22)$$

On the other hand, by noting that $u(t)$ is determined once the current state $(x(t), \dot{x}(t))$ is known, it can be argued using the principle of dynamic programming [35] that

$$\begin{aligned} & V(x(t), \dot{x}(t)) \\ &= \min_{u(t)} \left\{ \left(\frac{u(t)^2}{2} + \lambda_1 \right) dt + E[V(x(t+dt), \dot{x}(t+dt))|x(t), \dot{x}(t)] \right\} \end{aligned} \quad (23)$$

Using

$$V(x(t + dt), \dot{x}(t + dt)) = V(x(t), \dot{x}(t)) + dV \quad (24)$$

and taking conditional expectation, one obtains

$$\begin{aligned} E[V(x(t + dt), \dot{x}(t + dt)) | x(t), \dot{x}(t)] \\ = V(x, \dot{x}) + \frac{\partial V}{\partial x} \dot{x} dt + \frac{\partial V}{\partial \dot{x}} [u - \mu(x, \dot{x})] dt + \frac{1}{2} \frac{\partial^2 V}{\partial \dot{x}^2} \sigma^2 dt \end{aligned} \quad (25)$$

where the quantities on the RHS are evaluated at time t . Substituting into (23) and noting that $V(x, \dot{x})$ on both sides cancel out, one arrives at the Bellman's equation [35] that governs the optimal choice of $u(t)$ given state information at time t :

$$\min_u \left\{ \frac{\partial V}{\partial x} \dot{x} + \frac{\partial V}{\partial \dot{x}} [u - \mu(x, \dot{x})] + \frac{1}{2} \frac{\partial^2 V}{\partial \dot{x}^2} \sigma^2 + \frac{1}{2} u^2 + \lambda_1 \right\} = 0 \quad (26)$$

Solving for the minimum gives the optimal control force

$$u^*(x, \dot{x}) = -\frac{\partial V(x, \dot{x})}{\partial \dot{x}} \quad (27)$$

and the resulting Bellman's equation reads

$$\frac{\partial V}{\partial x} \dot{x} + \frac{\partial V}{\partial \dot{x}} [u - \mu(x, \dot{x})] + \frac{1}{2} \frac{\partial^2 V}{\partial \dot{x}^2} \sigma^2 - \frac{1}{2} \left(\frac{\partial V}{\partial \dot{x}} \right)^2 + \lambda_1 = 0 \quad (28)$$

Because of the way V is related to the control force, it is often called the 'control potential'. The Bellman's equation in (28) is a nonlinear partial differential equation (PDE) on the state-space domain $(-b_0, b_0) \times (-\infty, +\infty)$. Note that only the first two terms of J are reflected in (8). The third term of J , which is related to the exit velocity, is reflected in the boundary conditions for V :

$$V(b_0, \dot{x}) = -\lambda_2 \dot{x} \text{ for } \dot{x} > 0 \text{ and } V(-b_0, \dot{x}) = 0 \text{ for } \dot{x} < 0 \quad (29)$$

The Bellman's equation belongs to a class of PDEs that can often be solved numerically by iteration, despite its nonlinearity. We next present the numerical solution for V by means of a method called Jacobi iteration [34]. The partial derivatives are approximated by their finite difference on a mesh of the domain. For this purpose it is important to recognize the flow of information from the boundary conditions to affect the solution within the domain. For $\dot{x} > 0$, the right boundary $x = b_0$ is constrained and so information should flow leftward, suggesting a forward difference for $\partial V / \partial x$. Similarly, for $\dot{x} < 0$, V is constrained on the left boundary $x = -b_0$ and so a backward difference should be used for $\partial V / \partial x$. For $\dot{x} = 0$, $\partial V / \partial \dot{x}$ does not appear in the equation. By similar reasoning, we approximate $\partial V / \partial \dot{x}$ by forward, central and backward difference for $\dot{x} > 0$, $\dot{x} = 0$ and $\dot{x} < 0$, respectively.

Approximating the partial derivatives in (28) by their finite difference and minimizing, one obtains the optimal control force as the finite difference approximation of the control potential. The resulting Bellman's equation is

$$V_{i,j} = p_1 V_{i,j+1} + p_2 V_{i,j-1} + p_3 V_{i+1,j} + B_{i,j} (u_{i,j}^2 + \lambda_1) \quad (30)$$

where

$$p_k = p_k' / \sum_{r=1}^3 p_r', \quad B_{i,j} = 1 / \sum_{r=1}^3 p_r' \quad (31)$$

and,

$$p_1' = \frac{\sigma^2}{2\Delta\dot{x}_{i,j}^2}, \quad p_2' = \frac{\sigma^2}{2\Delta\dot{x}_{i,j}^2} - \frac{\hat{u}_{i,j} - \mu_{i,j}}{\Delta\dot{x}_{i,j}}, \quad p_3' = \frac{|\dot{x}_j|}{\Delta x_i} \text{ for } \dot{x}_{i,j} > 0 \quad (32)$$

$$p_1' = \frac{\sigma^2}{2\Delta\dot{x}_{i,j}^2} + \frac{\hat{u}_{i,j} - \mu_{i,j}}{2\Delta\dot{x}_{i,j}}, \quad p_2' = \frac{\sigma^2}{2\Delta\dot{x}_{i,j}^2} - \frac{\hat{u}_{i,j} - \mu_{i,j}}{2\Delta\dot{x}_{i,j}},$$

$$p_3' = 0 \text{ for } \dot{x}_{i,j} = 0 \quad (33)$$

$$p_1' = \frac{\sigma^2}{2\Delta\dot{x}_{i,j}^2} + \frac{\hat{u}_{i,j} - \mu_{i,j}}{\Delta\dot{x}_{i,j}}, \quad p_2' = \frac{\sigma^2}{2\Delta\dot{x}_{i,j}^2}, \quad p_3' = \frac{|\dot{x}_j|}{\Delta x_i} \text{ for } \dot{x}_{i,j} < 0 \quad (34)$$

We have dropped the dependence on (i, j) in p_k and p_k' to simplify notations. Note that $V_{i,j}$ in (30) cannot be solved explicitly because $\hat{u}_{i,j}$ depends on $V_{i,j}$. Nevertheless, the equation suggests solution for $V_{i,j}$ through the following iteration:

$$V_{i,j}^{(n+1)} = p_1^{(n)} V_{i,j+1}^{(n)} + p_2^{(n)} V_{i,j-1}^{(n)} + p_3^{(n)} V_{i+1,j}^{(n)} + B_{i,j}^{(n)} (\hat{u}_{i,j}^2 + \lambda_1) \quad (35)$$

where a superscript ' (n) ' denotes that the quantity is calculated using value at the n -th iteration. Convergence of this iterative scheme is ensured by requiring that $p_1^{(n)} + p_2^{(n)} + p_3^{(n)} = 1$ and $p_1^{(n)}, p_2^{(n)}, p_3^{(n)} > 0$, since then the mapping from the matrix $\underline{V}^{(n)} = [V_{i,j}^{(n)}]_{i,j}$ to $\underline{V}^{(n+1)} = [V_{i,j}^{(n+1)}]_{i,j}$ is a contraction. The first requirement is automatically satisfied by the definition of $p_k^{(n)}$. The second requirement can be enforced by limiting $\hat{u}_{i,j}$ to lie within $\left(\mu_{i,j}^{(n)} - \frac{\sigma^2}{\Delta\dot{x}}, \mu_{i,j}^{(n)} + \frac{\sigma^2}{\Delta\dot{x}}\right)$. The control law obtained in this manner is optimal only among the space of all those that make the iterative scheme convergent. The bounds on $\hat{u}_{i,j}$ are inactive when $\Delta\dot{x} \rightarrow 0$. This means that when the bounds are found to be active the converged solution is only suboptimal and it can be improved by refining the mesh on $\dot{x}$ (reducing $\Delta\dot{x}$).

The parameters λ_1 and λ_2 control the relative importance of the three objectives in the design of the optimal control law. In principle they can be chosen to maximize variance reduction, although such choice is not trivial and may not be worthwhile to pursue. For effective variance reduction it is sufficient to assign their values to

the right order of magnitude. As a simple choice, we may require the three terms in $E[J(u)|x, \dot{x}]$ to be of the same order of magnitude, which suggests

$$\lambda_1 = O\left(\frac{E}{t_0}\right) \text{ and } \lambda_2 = O\left(\frac{E}{\omega b_0}\right) \quad (36)$$

where E is the energy of the linear-elastic design point that pushes the structure from rest to the level $x = b$ at time t_0 . The time t_0 may be taken as some reasonable time at which the response is expected to go from the linear regime to yielding. Note that the linear-elastic design point can be obtained easily from the unit impulse response function. The assignment of λ_1 and λ_2 need not be precise, as the aim is to specify them just to the right order of magnitude.

3.2 Double Barrier Problem

For single-barrier problems, e.g., up-crossing failure, one can use an adapted process designed for up-crossing and then apply importance sampling using fixed point. For double-barrier crossing problems, one can use an average of ISD based on the up-crossing and down-crossing adapted process to account for failures due to up-crossing and down-crossing, respectively:

$$f(\mathbf{z}) = \frac{1}{2}\phi(\mathbf{z} - \hat{\mathbf{z}}^+) + \frac{1}{2}\phi(\mathbf{z} - \hat{\mathbf{z}}^-) \quad (37)$$

where $\hat{\mathbf{z}}_+$ and $\hat{\mathbf{z}}_-$ denote the adapted process for up- and down-crossing respectively. By symmetry the up-crossing and down-crossing adapted process in the time domain are given by:

$$u^+ = -\frac{\partial V(x, \dot{x})}{\partial \dot{x}}, \quad u^- = \frac{\partial V(-x, -\dot{x})}{\partial \dot{x}} \quad (38)$$

Using the ISD in (37), the failure probability is given by

$$P_F = 2E_f [I(\mathbf{Z}' \in F)R(\mathbf{Z}')] \quad (39)$$

where the expectation is taken with $\mathbf{Z}'$ distributed according to f and

$$R(\mathbf{z}') = \left[\frac{1}{2} \frac{\phi(\mathbf{z}' - \hat{\mathbf{z}}^+)}{\phi(\mathbf{z}')} + \frac{1}{2} \frac{\phi(\mathbf{z}' - \hat{\mathbf{z}}^-)}{\phi(\mathbf{z}')} \right]^{-1} \quad (40)$$

Evaluating the expectation in (39) to account for the bimodal nature of f gives

$$P_F = E_+ [I(\mathbf{Z}' \in F)R(\mathbf{Z}')] + E_- [I(\mathbf{Z}' \in F)R(\mathbf{Z}')] = 2E_+ [I(\mathbf{Z}' \in F)R(\mathbf{Z}')] \quad (41)$$

where the subscript ‘+’ in the first expectation denotes that $\mathbf{Z}'$ is distributed as $\phi(\mathbf{z} - \hat{\mathbf{z}}^+)$; similar notation applies for the second expectation. The second equality makes use of the observation that the up-crossing and down-crossing expectations are the same, due to symmetry.

4 Numerical Investigation

Consider an SDOF elasto-plastic structure with $\omega = 2\pi$, $\zeta = 1\%$ and yield displacement $b_0 = 1$. Failure is defined as first passage over the double barrier $x = \pm b_F$, $b_F = 3$, within a duration of interest $t_F = 20$ s. The spectral intensity of the white noise is assumed to be $S = 0.86$, for which the response standard deviation at 20 s is approximately equal to 1.

4.1 Control Law

We first study the characteristics of the control law. Here we set $\lambda_1 = \lambda_2 = 2$, which is found to yield a reasonable failure rate. Other values of λ_1 and λ_2 shall be considered later when variance reduction is investigated. Figure 1 shows the contour of the control potential $V(x, \dot{x})$ and the corresponding control law $u(x, \dot{x})$. These are computed by Jacobi iteration on a 40×400 $(x, \dot{x})$ -grid shown in the figure. As enforced by the boundary conditions, V is identically zero along the lower-left boundary and it decreases with $\dot{x}$ along the upper-right boundary. It is seen that V behaves differently in the upper and lower quadrants, partly due to the different boundary

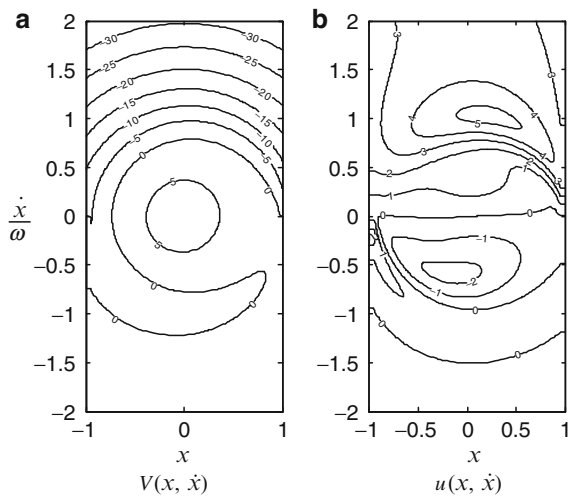


Fig. 1 Contour of control potential $V(x, \dot{x})$ and control force $u(x, \dot{x})$

conditions imposed in the first and third quadrant. In the lower quadrants (third and fourth) it is essentially zero, while in the upper quadrants (first and second) it roughly decays radially with x and $\dot{x}/\omega$, which is akin to the total energy. The control law $u(x, \dot{x})$ shown in Fig. 1b, which is just $-\partial V/\partial \dot{x}$, shows a peak in the upper quadrants and a trough in the lower quadrants, although the former is more pronounced. The contour shows that the control law is not directly proportional to $\dot{x}$. It does not increase monotonically with $\dot{x}$ even within the increasing region the rate is not constant. This suggests that a constant gain feed-forward rule can be far from being optimal. When $\dot{x}$ is small the control law is small regardless of x . This is the region where the stochastic effect of the white noise dominates and so saving control effort is a good strategy.

We next investigate the manner in which the control law drives the response to first passage failure. As a reference Fig. 2 shows the design point excitation that targets to drive the response to first passage failure at 20 s, computed using an efficient algorithm documented in [27]. The design point excitation initially grows the response by resonance. It then applies positive pulses to create positive plastic displacements that accumulate until failure while avoiding negative plastic displacements. The design point excitation is effective in generating large response only in the absence of white noise. During importance sampling where the excitation consists of the design point excitation and white noise, previous studies reported that its actions are not effective, leading to a small failure rate [29].

Figures 3 and 4 show two typical samples of the control force $u(t)$, the corresponding response $x(t)$ and hysteretic history of restoring force. Note that the response $x(t)$ is due to the combined action of the control force and white noise (not shown). The samples of control force exhibit two properties that are important to generating response effectively: (1) resonance and (2) asymmetry that avoids negative plastic displacements. Their adapted stochastic nature promotes synchronization with the response and maintains their effectiveness.

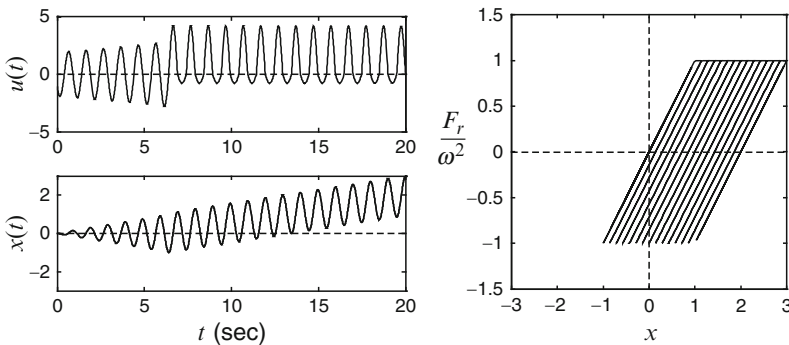


Fig. 2 Design point excitation (first passage failure at 20 s)

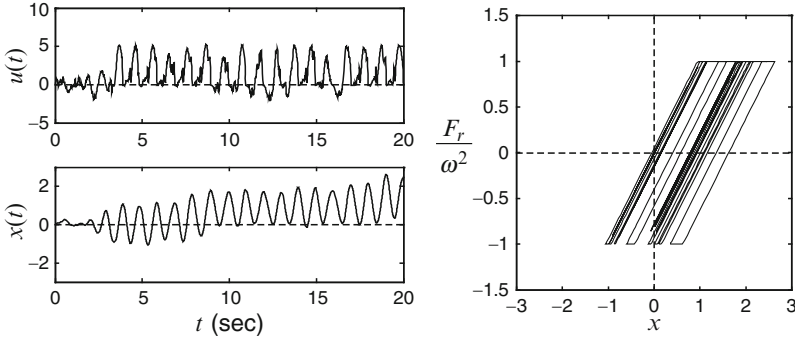


Fig. 3 Random sample of $u(t)$ and corresponding $x(t)$ (not failed)

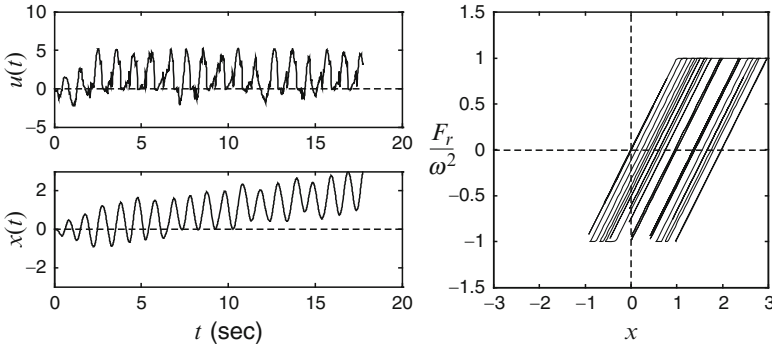


Fig. 4 Random sample of $u(t)$ and corresponding $x(t)$ (failed at 17.5 s)

4.2 Variance Reduction

We first investigate the robustness of the control force (for the purpose of importance sampling) with respect to the choice of λ_1 and λ_2 . We consider different combinations of their values and perform importance sampling in each case. The combinations covered $\lambda_1 = 1, 1.5$ and $\lambda_2 = 1.5, 2, 2.5$. In all cases the contour of control law was found to be qualitatively similar. The unit c.o.v. was estimated with one hundred thousand (100,000) samples. The results are summarized in Table 1. For each combination of λ_1 and λ_2 the values of the unit c.o.v. Δ , failure rate Q_F and conditional c.o.v. $\Delta_{R|F}$ are shown. Variance reduction is assessed in term of the unit c.o.v. Δ , whereas its mechanism can be analyzed with Q_F and $\Delta_{R|F}$, through the identity in (8). Here, the values of λ_1 and λ_2 chosen, in multiples of their respective scales, are of the order of 1. The values chosen are intended to illustrate the effect of λ_1 and λ_2 on efficiency, and the trade-off between trade-off of Q_F and $\Delta_{R|F}$ that possibly leads to a minimum Δ .

As shown in Table 1, the failure rate Q_F increases with both λ_1 and λ_2 . This is expected because in the objective function λ_1 reflects the importance of the time to

Table 1 Variance reduction for different choice of λ_1 and λ_2

		$\lambda_2 (\times E/\omega b_0)$		
$Q_F, \Delta, \Delta_{R F}$		1.5	2	2.5
$\lambda_1 (\times E/t_0)$	1	5%, 9.8, 2.0	12%, 9.7, 3.2	24%, 13.6, 6.6
	1.5	9%, 9.0, 2.5	21%, 7.6, 3.4	38%, 12.8, 7.9

Table 2 Previous reports on variance reduction

Method	Failure rate Q_F	Unit c.o.v. Δ	Conditional c.o.v. $\Delta_{R F}$
Direct MCS	2.4×10^{-5}	204	0
IS-design point [29]	5%	38	8.4
IS-adapted [33]	14%	20	7.4

next yield, while λ_2 reflects the importance of exit velocity. The higher the value of λ_1 , the more important it is to shorten the time to next yield and hence the higher the power the control law will spend. On the other hand, a high value of λ_2 targets a high exit velocity, which again increases the power.

The conditional c.o.v. $\Delta_{R|F}$ increases with the failure rate because it roughly increases with the energy of the adapted process (control force). The rate of increase is not uniform and can be quite drastic for high failure rates. The net effect is that variance reduction is better achieved with a balance between the failure rate and the conditional c.o.v. As shown in Table 1, the value of unit c.o.v. is similar for different combinations of λ_1 and λ_2 . No optimization with regard to (λ_1, λ_2) is intended in this work, although it is observed that variance reduction is highest around $\lambda_1 = 1.5$ and $\lambda_2 = 2$. It is more important for the variance reduction capability to be robust to the choice of (λ_1, λ_2) than to locate the optimal value of (λ_1, λ_2) because the optimum is likely to be problem dependent.

The unit c.o.v. in other range of values of λ_1 and λ_2 can be expected from their roles in the method. Essentially, as their values initially increase from zero they tend to decrease Δ because in that range they can increase the failure rate significantly while overweigh the accompanying increase in $\Delta_{R|F}$. At the other extreme if they are very large too much effort is spent on driving the response to plastic excursions and their large energy will lead to exponentially large $\Delta_{R|F}$. The resulting detrimental increasing effect on Δ can hardly be compensated by the increase in the failure rate Q_F . The range of values of λ_1 and λ_2 shown in the Table 1 is the ‘interesting’ region where the effects of Q_F and $\Delta_{R|F}$ are trading off.

We next investigate the variance reduction capability of the proposed importance sampling method. As a reference the failure probability has been estimated by direct Monte Carlo with one million samples to be 2.4×10^{-5} (c.o.v. = 20%). The failure probability has been previously estimated by importance sampling with design points [29] and importance sampling with adapted process designed heuristically with updating at stationary points [33]. Table 2 summarizes the reported variance reduction efficiency.

The values of unit c.o.v. reported in Table 1 are generally smaller than those reported previously, reflecting a progress in variance reduction. In particular, for

a similar failure rate the conditional c.o.v. of the proposed method is significantly smaller than that of the adapted process developed in the previous work ([33], last row), demonstrating the effectiveness of the stochastic control law. To make a comparison in terms of computational efforts, suppose a $\delta = 30\%$ c.o.v. in the target failure probability of 2.4×10^{-5} is desired. This means the average number of samples required is $N = \Delta^2 / \delta^2 \sim 10\Delta^2$. Based on the values of Δ reported in Tables 1 and 2, the number of samples for direct MCS, importance sampling using design points [29], importance sampling using previous heuristic rules [33] and the present work are 416×10^3 , 14×10^3 , 4×10^3 and 1×10^3 . Here, for the present work we have used a representative value of $\Delta = 10$. These numbers show a progressive improvement in variance reduction using importance sampling technique.

Finally, the computational efforts required by different methods on the problem are summarized in Fig. 5, which shows the number of samples required to achieve a c.o.v. of 50% in the failure probability estimate. As a baseline the computational effort required by direct Monte Carlo (straight line) grows in an exponential manner with decreasing failure probability. Since direct Monte Carlo is the most robust method, any method that performs to the right of the line, and hence is less efficient, should be disposed of. The scattered dots show the performance of methods that make use of Markov Chain Monte Carlo (MCMC), e.g., Subset Simulation and line-sampling, as reported in Schueller & Pradlwarter [15]. Along the effort on importance sampling for elasto-plastic structures, the triangles show the performance using design points [29], which have similar efficiency as methods using MCMC. In an overall sense it is inferior to methods using MCMC because it is less robust. The squares and the stars show the performance using adapted process constructed based on heuristic rules and stochastic control theory, respectively. They show a progressive improvement in efficiency, although not as drastic as importance sampling for linear structures.

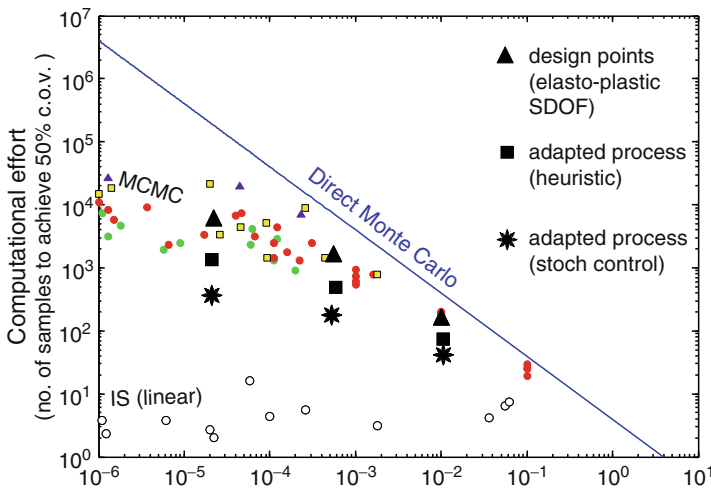


Fig. 5 Summary of computational effort required by different methods

5 Conclusions

An importance sampling method based on adapted process has been presented. There is great flexibility in the design of the adapted process, although a stochastic control approach has been adopted in this work. While the ideal design goal for the purpose of solving the first passage problem is not clear, it has been set heuristically and seemed to work well. Given such goals, the actual algorithm that generates the adapted process as a stochastic process is derived following a stochastic control approach. The objective function reflects the expected energy needed to next yield as well as the exit velocity. This objective is heuristic but it simplifies design of the controller. Determining the optimal controller involves solution of a control potential that satisfies the Bellman's equation. The Bellman's equation is a nonlinear PDE on the state-space of response, and it is numerically solved by Jacobi iteration method. When the control potential cannot be solved analytically, as is often the case, it implies that the method will suffer from the curse of the state-space dimension for MDOF structures. In view of this, a viable strategy is to use control law that only depend on partial observation of the response state. For example, the unobserved states of the structure may be estimated by filtering techniques in order to allow the Bellman equation to be written in terms of the observable states, which can be of much lower dimension than the full states of a general MDOF structure. The resulting control law will be suboptimal to the one that uses full-state information, but its design is much more manageable. After all, by speculation a good design that can systematically cope with the complexity of the hysteretic phenomenon suffices to bring out the merit of importance sampling.

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References

1. Wen, Y.K.: Method for random vibration of hysteric systems. *J. Eng. Mech.* **102**, 249–263 (1976)
2. Soong, T.T., Grigoriu, M.: *Random Vibration of Mechanical and Structural Systems*. Prentice-Hall, NJ (1993)
3. Lin, Y.K., Cai, G.Q.: *Probabilistic Structural Dynamics: Advanced Theory and Applications*. McGraw-Hall, New York (1995)
4. Lutes, L.D., Sarkani, S.: *Stochastic Analysis of Structural and Mechanical Vibrations*. Prentice Hall, NJ (1997)
5. Schueller, G.I.: Developments in stochastic structural mechanics. *Arch. Appl. Mech.* **75**, 755–773 (2006)
6. Schueller, G.I., Pradlwarter, H.J., Koutsourelakis, P.S.: A critical appraisal of reliability estimation procedures for high dimensions. *Probab. Eng. Mech.* **19**, 463–74 (2004)
7. Au, S.K., Beck, J.L.: Importance sampling in high dimensions. *Struct. Saf.* **25**(2), 139–163 (2003)

8. Rubinstein, R.Y.: *Simulation and the Monte-Carlo Method*. Wiley, New York (1981)
9. Hammersley, J.M., Handscomb, D.C.: *Monte-Carlo Methods*. Methuen, London (1964)
10. Hohenbichler, M., Rackwitz, R.: Improvement of second-order reliability estimates by importance sampling. *J. Eng. Mech. ASCE* **114**(12), 2195–2198 (1988)
11. Melchers, R.E.: Importance sampling in structural systems. *Struct. Saf.* **6**, 3–10 (1989)
12. Bucher: Adaptive sampling – an iterative fast Monte Carlo procedure. *Struct. Saf.* **5**, 119–126 (1988)
13. Melchers, R.E.: Search-based importance sampling. *Struct. Saf.* **9**, 117–128 (1990)
14. Ang, G.L., Ang, A.H.S., Tang, W.H.: Optimal importance sampling density estimator. *J.Eng. Mech. ASCE* **118**(6), 1146–1163 (1992)
15. Schueller, G.I., Pradlwarter, H.J.: Benchmark study on reliability estimation in higher dimensions of structural systems – an overview. *Struct. Saf.* **29**(3), 167–182 (2007)
16. Au, S.K., Ching, J., Beck, J.L.: Application of subset simulation methods to reliability benchmark problems. *Struct. Saf.* **29**(3), 183–193 (2007)
17. Pradlwarter, H.J., Schueller, G.I., Koutsourelakis, P.S., et al.: Application of line sampling simulation method to reliability benchmark problems. *Struct. Saf.* **29**(3), 208–221 (2007)
18. Au, S.K., Beck, J.L.: First excursion probability for linear systems by very efficient importance sampling. *Probab. Eng. Mech.* **16**(3), 193–207 (2001)
19. Yuen, K.V., Katafygiotis, L.S.: An efficient simulation method for reliability analysis of linear dynamical systems using simple additive rules of probability. *Probab. Eng. Mech.* **20**, 109–114 (2005)
20. Jensen, H.A., Valdebenito, M.A.: Reliability analysis of linear dynamical systems using approximate representations of performance functions. *Struct. Saf.* **29**(3), 222–237 (2007)
21. Iyengar, R.N.: Worst inputs and a bound on the highest peak statistics of a class of non-linear systems. *J. Sound Vib.* **25**(1), 29–37 (1972)
22. Koo, H., Kiureghian, D., Fujimura, K.: Design-point excitation for non-linear random vibrations. *Probab. Eng. Mech.* **20**(2), 136–147 (2005)
23. Westermo, B.D.: The critical excitation and response of simple dynamic systems. *J. Sound Vib.* **100**(2), 233–242 (1985)
24. Roberts, J.B., Spanos, P.D.: *Random Vibration and Statistical Linearization*. Wiley, New York (1990)
25. Drenick, R.F.: The critical excitation of nonlinear systems. *J. Appl. Mech.* **44**(E2), 333–336 (1977)
26. Takewaki, I.: Critical excitation for elastic-plastic structures via statistical equivalent linearization. *Probab. Eng. Mech.* **17**(1), 73–84 (2002)
27. Au, S.K.: Critical excitation of SDOF elasto-plastic systems. *J. Sound Vib.* **296**(4–5), 714–733 (2006)
28. Au, S.K.: Sub-critical excitations of SDOF elasto-plastic systems. *J. Nonlinear Mech.* **41**(9), 1095–1108 (2006)
29. Au, S.K., Lam, H.F., Ng, C.T.: Reliability analysis of SDOF elasto-plastic systems: Part I: critical excitation. *J. Eng. Mech.* **133**(10), 1072–1080 (2007)
30. Girsanov, I.V.: On transforming a certain class of stochastic processes by absolutely continuous substitution of measures. *Theory Probab. Appl.* **5**, 285–301 (1960)
31. Engelund, S., Rackwitz, R.: A benchmark study on importance sampling techniques in structural reliability. *Struct. Saf.* **12**, 255–76 (1993)
32. Protter, P.: *Stochastic Integration and Differential Equations*. Springer-Verlag, Berlin (1990)
33. Au, S.K.: First passage probability of elasto-plastic systems by importance sampling with adapted processes. *Probab. Eng. Mech.* **23**(2–3), 114–124 (2008)
34. Kushner, H.J., Dupuis, P.: *Numerical Methods for Stochastic Control Problems in Continuous Time*, 2nd edn. Springer, New York (2001)
35. Kushner, H.J.: *Stochastic Stability and Control*. Academic Press, New York/London (1967)

Use of Time-Variant Spectral Characteristics of Nonstationary Random Processes in the First-Passage Problem for Earthquake Engineering Applications

Michele Barbato

Abstract In this work, the spectral characteristics of non-stationary random processes are applied to the time-variant first-passage problem in structural reliability. The first-passage problem consists in computing the probability of a response quantity exceeding a deterministic threshold in a given interval of time when the parameters defining the structural system and/or the loading are random quantities. The Vanmarcke and modified Vanmarcke approximations of the failure probability are expressed herein as integrals in time of the closed-forms of the corresponding hazard functions. These closed-forms refer to linear elastic systems subjected to a time-modulated coloured noise base excitation and are obtained using the corresponding closed-form solutions for the time-variant bandwidth parameter of the system response processes. These approximate solutions to the first-passage problem are compared with the classical Poisson approximation and Monte Carlo simulation results for an idealized linear elastic model of a three-story one-bay shear-type steel building. The retrofit of this benchmark structure with viscous dampers is also considered, allowing (1) to illustrate the use of the newly available closed-form approximations of the failure probability for non-classically damped linear elastic systems and (2) to show an example of practical use in structural engineering of the presented analytical derivations. The earthquake base excitation is modelled as a filtered white noise process defined by the well-known Kanai-Tajimi equation and time-modulated by the widely-used Shinozuka and Sato modulating function. The failure condition is expressed in terms of interstory drifts outcrossing specified deterministic thresholds. The results presented in this study show that the two Vanmarcke approximations can improve considerably the estimates of the failure probability for the first-passage problem when compared with the simpler Poisson approximation.

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1 Introduction

The dynamic behaviour of structural and mechanical systems subjected to uncertain dynamic excitations can be described, in general, through random processes. The probabilistic characterisation of these random processes can be extremely complex, when non-stationary and/or non-Gaussian input processes are involved. However, specific applications may require only an incomplete description of the considered stochastic processes, based on the process spectral characteristics [1, 2]. In particular, the so-called non-geometric spectral characteristics (*NGSCs*) [3–6] can be employed to evaluate the time-variant central frequency and bandwidth parameters, which characterise in a synthetic way a non-stationary stochastic process (*NSSP*). The *NGSCs* have been proved appropriate for describing *NSSPs* and can be effectively employed in structural reliability applications, such as the computation of the time-variant probability that a random process out-crosses a given limit-state threshold [7].

In this work, exact closed-form analytical solutions for the *NGSCs* of *NSSPs* representing the response of single- and multi-degree-of-freedom (*SDOF/MDOF*) linear elastic structural models subjected to time-modulated coloured noises are illustrated and employed. These solutions were derived and presented in a previous work by the author [6], based on the extension of *NGSCs* to complex-values stochastic processes [5]. These *NGSC* closed-form solutions correspond to the following significant advances in random vibration theory: (1) the presented solutions are in analytical form, while previous exact solutions required numerical solution of integral expressions [7–9]; (2) the input process is generalized from a time-modulated white noise to a time-modulated coloured noise; and (3) the structural models considered here include not only proportionally damped but also non-proportionally damped *SDOF/MDOF* linear elastic systems. The *NGSCs* are used in this study to compute exactly and in closed-form the time-variant central frequency and bandwidth parameter of the response processes of both classically and non-classically damped *MDOF* linear elastic systems subjected to earthquake base excitations modelled as time-modulated coloured noise models. These closed-form solutions are then used to compute the classical and modified Vanmarcke approximations of the first-passage problem failure probability [10–12], with particular focus on earthquake engineering applications. For the sake of simplicity and without loss of generality, all random processes considered in this study are zero-mean processes. For these processes, the auto- and cross-covariance functions coincide with their auto- and cross-correlation functions, respectively.

2 Central Frequency and Bandwidth Parameters of Non-Stationary Stochastic Processes

A non-stationary stochastic process (*NSSP*) $X(t)$ can be expressed in the general form of a Fourier-Stieltjes integral as [1]

$$X(t) = \int_{-\infty}^{\infty} A_X(\omega, t) \cdot e^{j\omega t} \cdot dZ(\omega) \quad (1)$$

in which t = time, ω = frequency parameter, $j = \sqrt{-1}$, $A_X(\omega, t)$ = complex-valued deterministic time-frequency modulating function, and $dZ(\omega)$ = zero-mean orthogonal-increment process defined so that

$$E[dZ^*(\omega_1) \cdot dZ(\omega_2)] = \Phi(\omega_1) \cdot \delta(\omega_1 - \omega_2) \cdot d\omega_1 d\omega_2 \quad (2)$$

where $E[\cdot]$ = mathematical expectation, $\Phi(\omega)$ = power spectral density (*PSD*) function of the embedded stationary process $X_S(t)$, defined as

$$X_S(t) = \int_{-\infty}^{\infty} e^{j\omega t} \cdot dZ(\omega) \quad (3)$$

$\delta(\cdot)$ = Dirac delta function, and the superscript $(\cdot)^*$ denotes the complex-conjugate operator. The process $X(t)$ has the following evolutionary power spectral density (*EPSD*) function:

$$\Phi_{XX}(\omega, t) = A_X^*(\omega, t) \cdot \Phi(\omega) \cdot A_X(\omega, t) \quad (4)$$

It is also convenient to define the process $Y(t)$ as the modulation (with modulating function $A_X(\omega, t)$) of the stationary process $Y_S(t)$ defined as the Hilbert transform of the embedded stationary process $X_S(t)$ [13, 14] i.e.,

$$Y(t) = -j \int_{-\infty}^{\infty} \text{sign}(\omega) \cdot A_X(\omega, t) \cdot e^{j\omega t} \cdot dZ(\omega) \quad (5)$$

For each *NSSP* $X(t)$, two sets of non-geometric spectral characteristics (*NGSCs*) can be defined as follows [5]

$$\begin{cases} c_{ik,XX}(t) &= \int_{-\infty}^{\infty} \Phi_{X^{(i)}X^{(k)}}(\omega, t) \cdot d\omega = C_{X^{(i)}X^{(k)}}(t) \\ c_{ik,XY}(t) &= \int_{-\infty}^{\infty} \Phi_{X^{(i)}Y^{(k)}}(\omega, t) \cdot d\omega = C_{X^{(i)}Y^{(k)}}(t) \end{cases} \quad (6)$$

where $C_{X^{(i)}X^{(k)}}(t)$ = cross-covariance of random processes $X^{(i)}(t)$ and $X^{(k)}(t)$, and $C_{X^{(i)}Y^{(k)}}(t)$ = cross-covariance of random processes $X^{(i)}(t)$ and $Y^{(k)}(t)$, in which

$$W^{(m)}(t) = \frac{d^m W(t)}{dt^m}, \quad W = X, Y; \quad m = i, k \quad (7)$$

The evolutionary cross-PSD functions $\Phi_{X^{(i)}W^{(k)}}(\omega, t)$ ($W = X, Y$ and $i, k = 0, 1, \dots$) are given by

$$\Phi_{X^{(i)}W^{(k)}}(\omega, t) = A_{X^{(i)}}^*(\omega, t) \cdot \Phi(\omega) \cdot A_{W^{(k)}}(\omega, t) \quad (8)$$

where [5]

$$A_{W^{(i)}}(\omega, t) = e^{-j\omega t} \frac{\partial^i}{\partial t^i} [A_W(\omega, t) \cdot e^{j\omega t}] \quad (9)$$

Herein, it is assumed that the time-derivative processes in Eq. 6 exist in the mean-square sense. In the particular case when $i = k = n$, the cross-covariance in Eq. 6₁ reduces to the variance of the n th time-derivative of the process $X(t)$, i.e., $C_{X^{(n)}X^{(n)}}(t) = \sigma_{X^{(n)}}^2(t)$.

The four *NGSCs* $c_{00,XX}(t)$, $c_{11,XX}(t)$, $c_{01,XX}(t)$ and $c_{01,XY}(t)$ are particularly relevant to random vibration theory and time-variant reliability applications. In fact, $c_{00,XX}(t)$ and $c_{11,XX}(t)$ represent the variance of the process and of its first time-derivative (i.e., $\sigma_X^2(t)$ and $\sigma_{\dot{X}}^2(t)$), respectively, $c_{01,XX}(t)$ denotes the cross-covariance of the process and its first time derivative (i.e., $C_{X\dot{X}}(t)$), and $c_{01,XY}(t)$ represents the cross-covariance of the process $X(t)$ and the first time-derivative of the process $Y(t)$ (i.e., $C_{X\dot{Y}}(t)$). Notice that the definitions in Eq. 6 for $c_{00,XX}(t)$, $c_{11,XX}(t)$, $c_{01,XX}(t)$ and $c_{01,XY}(t)$ are valid for both real-valued and complex-valued *NSSPs* [5]. In the case of real-valued *NSSPs*, these definitions are equivalent to the one proposed in [3,4]. The *NGSCs* $c_{00,XX}(t)$, $c_{11,XX}(t)$ and $c_{01,XY}(t)$ are used in the definition of the time-variant central frequency, $\omega_c(t)$, and bandwidth parameter, $q(t)$, of the *NSSP* $X(t)$ as [5]

$$\omega_c(t) = \frac{c_{01,XY}(t)}{c_{00,XX}(t)} = \frac{C_{X\dot{Y}}(t)}{\sigma_X^2(t)} \quad (10)$$

$$q(t) = \left(1 - \frac{c_{01,XY}^2(t)}{c_{00,XX}(t) \cdot c_{11,XX}(t)} \right)^{0.5} = \left(1 - \frac{C_{X\dot{Y}}^2(t)}{\sigma_X^2(t) \cdot \sigma_{\dot{X}}^2(t)} \right)^{0.5} \quad (11)$$

The time-variant central frequency and bandwidth parameter are useful in describing the time-variant spectral properties of a real-valued *NSSP* $X(t)$. The central frequency $\omega_c(t)$ provides the characteristic/predominant frequency of the process at each instant of time. The bandwidth parameter $q(t)$ provides information on the spectral bandwidth of the process at each instant of time. Notice that a *NSSP* can behave as a narrowband and a broadband process at different instants of time. For complex-valued *NSSPs*, the complex-valued central frequency and bandwidth parameter defined in Eqs. 10 and 11 lose the simple physical interpretation available for real-valued *NSSPs*, even though the computation of the corresponding *NGSCs* is instrumental to the solution of problems requiring a state-space representation. In addition, the bandwidth parameter $q(t)$ plays an important role in time-variant reliability analysis, since it is an essential ingredient of analytical approximations to the time-variant failure probability for the first-passage reliability problem [11, 12, 15–19].

3 Spectral Characteristics of the Stochastic Response of SDOF/MDOF Linear Systems Subjected to Non-Stationary Excitations

In this section, the spectral characteristics of the response processes of SDOF/MDOF linear systems subjected to non-stationary excitations are introduced. First, complex modal analysis is briefly reviewed and the complex modal response processes are computed. Then, the spectral characteristics of the (real-valued) response processes of linear systems are obtained from the spectral characteristics of the complex modal response processes. Finally, the closed-form solutions for the spectral characteristics are presented for linear elastic systems subjected to time-modulated coloured noise excitations.

3.1 Complex Modal Analysis

A state-space formulation of the equations of motion for *MDOF* linear systems is useful to describe the response of both classically and non-classically damped systems [20]. The general (second-order) equations of motion for an n -degree-of-freedom linear system are, in matrix form,

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{K}\mathbf{U}(t) = \mathbf{P}F(t) \quad (12)$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K} = n \times n$ time-invariant mass, damping and stiffness matrices, respectively; $\mathbf{U}(t)$ = length- n vector of nodal displacements, $\mathbf{P}$ = length- n load distribution vector, $F(t)$ = scalar function describing the time-history of the external loading (random process), and a superposed dot denotes differentiation with respect to time. The matrix equation of motion, Eq. 12, can be recast into the following first-order matrix equation

$$\dot{\mathbf{Z}}(t) = \mathbf{G}\mathbf{Z}(t) + \tilde{\mathbf{P}}F(t) \quad (13)$$

where

$$\mathbf{Z}(t) = \begin{bmatrix} \mathbf{U}(t) \\ \dot{\mathbf{U}}(t) \end{bmatrix}_{(2n \times 1)} \quad (14)$$

$$\mathbf{G} = \begin{bmatrix} \mathbf{0}_{n \times n} & \mathbf{I}_{n \times n} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}_{(2n \times 2n)} \quad (15)$$

$$\tilde{\mathbf{P}} = \begin{bmatrix} \mathbf{0}_{n \times 1} \\ \mathbf{M}^{-1}\mathbf{P} \end{bmatrix}_{(2n \times 1)} \quad (16)$$

The subscripts in Eqs. 14–16 indicate the dimensions of the vectors and matrices to which they are attached. Using the complex modal matrix $\mathbf{T}$ formed from the complex eigenmodes of matrix $\mathbf{G}$, the first-order matrix equation given in Eq. 13 can be decoupled into the following $2n$ normalised complex modal equations

$$\dot{S}_i(t) = \lambda_i \cdot S_i(t) + F(t), \quad i = 1, 2, \dots, 2n \quad (17)$$

where $\mathbf{S} = [S_1(t) \ S_2(t) \ \dots \ S_{2n}(t)]^T$ = normalised complex modal response vector, λ_i ($i = 1, \dots, 2n$) = complex eigenvalues of the system matrix $\mathbf{G}$, and the super-script $(\cdot)^T$ denotes the matrix transpose operator. The response of the linear *MDOF* system can be obtained as

$$\mathbf{Z}(t) = \mathbf{T}\mathbf{F}\mathbf{S}(t) = \tilde{\mathbf{T}}\mathbf{S}(t) \quad (18)$$

in which $\mathbf{F}$ = diagonal matrix containing the $2n$ modal participation factors Γ_i , defined as the i th component of vector $\mathbf{T}^{-1}\tilde{\mathbf{P}} = [\Gamma_1 \ \Gamma_2 \ \dots \ \Gamma_{2n}]^T$, and $\tilde{\mathbf{T}} = \mathbf{T}\mathbf{F}$ = effective modal participation matrix. Assuming that the system is initially at rest, the solution of Eq. 17 can be expressed by the following Duhamel integral:

$$S_i(t) = \int_0^t e^{\lambda_i \cdot (t-\tau)} \cdot F(\tau) \cdot d\tau, \quad i = 1, 2, \dots, 2n \quad (19)$$

It is worth mentioning that the normalised complex modal responses $S_i(t)$, $i = 1, 2, \dots, 2n$, are complex conjugate by pairs. In the case of a non-stationary loading process, the loading function $F(t)$ can be expressed in general as (see Eq. 1)

$$F(t) = \int_{-\infty}^{\infty} A_F(\omega, t) \cdot e^{j\omega t} \cdot dZ(\omega) \quad (20)$$

It can be shown that the normalised complex modal responses are given by

$$S_i(t) = \int_{-\infty}^{\infty} A_{S_i}(\omega, t) \cdot e^{j\omega t} \cdot dZ(\omega), \quad i = 1, 2, \dots, 2n \quad (21)$$

where

$$A_{S_i}(\omega, t) = \int_{-\infty}^t e^{(\lambda_i - j\omega) \cdot (t-\tau)} \cdot A_F(\omega, \tau) \cdot d\tau, \quad i = 1, 2, \dots, 2n \quad (22)$$

3.2 NGSCs of Response Processes of MDOF Linear Systems Using Complex Modal Analysis

The state-space formulation of the equations of motion is also advantageous for the computation of the NGSCs of response processes of both classically and non-classically damped *MDOF* linear systems. If only Gaussian input processes are considered, only few spectral characteristics are needed to fully describe the response

processes of *MDOF* linear elastic systems, since the response processes are also Gaussian. In particular, if $U_i(t)$ denotes the i th *DOF* displacement response process of an *MDOF* linear elastic system subjected to Gaussian excitation, the only spectral characteristics required, e.g., for reliability applications, are ($i = 1, 2, \dots, n$)

$$\begin{cases} c_{00,U_i U_i}(t) = \sigma_{U_i}^2(t) \\ c_{11,U_i U_i}(t) = \sigma_{\dot{U}_i}^2(t) \\ c_{01,U_i U_i}(t) = C_{U_i \dot{U}_i}(t) \\ c_{01,U_i \dot{U}_i}(t) = C_{U_i \dot{U}_i}(t) \end{cases} \quad (23)$$

where $\dot{U}_i$ is the first time-derivative of the process U_i defined as (see Eq. 5 and [13, 14])

$$\dot{U}_i(t) = -j \int_{-\infty}^{\infty} \text{sign}(\omega) \cdot A_{U_i}(\omega, t) \cdot e^{j\omega t} \cdot dZ(\omega), \quad i = 1, 2, \dots, n \quad (24)$$

and $A_{U_i}(\omega, t)$ = time-frequency modulating function of the process $U_i(t)$. Similarly to the response processes (see Eq. 14), the following auxiliary state vector process can be defined

$$\boldsymbol{\Xi} = \begin{bmatrix} \mathcal{U} \\ \dot{\mathcal{U}} \end{bmatrix}_{(2n \times 1)} \quad (25)$$

Using complex modal decomposition, the cross-covariance matrices of the response processes and the auxiliary processes can be computed as

$$E[\mathbf{Z}(t)\mathbf{Z}^T(t)] = \tilde{\mathbf{T}}^* E[\mathbf{S}^*(t)\mathbf{S}^T(t)] \tilde{\mathbf{T}}^T \quad (26)$$

$$E[\mathbf{Z}(t)\boldsymbol{\Xi}^T(t)] = \tilde{\mathbf{T}}^* E[\mathbf{S}^*(t)\boldsymbol{\Sigma}^T(t)] \tilde{\mathbf{T}}^T \quad (27)$$

where the components of the vector process $\boldsymbol{\Sigma} = [\Sigma_1(t) \ \Sigma_2(t) \ \dots \ \Sigma_{2n}(t)]^T$ are defined as

$$\Sigma_i(t) = -j \int_{-\infty}^{\infty} \text{sign}(\omega) \cdot A_{S_i}(\omega, t) \cdot e^{j\omega t} \cdot dZ(\omega), \quad i = 1, 2, \dots, n \quad (28)$$

Equations 26 and 27 show that all quantities in Eq. 23 can be computed from the following spectral characteristics of complex-valued non-stationary processes ($i, m = 1, 2, \dots, 2n$)

$$\begin{cases} E[S_i^*(t) \cdot S_m(t)] = C_{S_i S_m}(t) \\ E[S_i^*(t) \cdot \Sigma_m(t)] = C_{S_i \Sigma_m}(t) \end{cases} \quad (29)$$

Notice also that knowledge of the spectral characteristics in Eq. 29 allows computation of the zero-th to second-order spectral characteristics of the components of any vector response quantity $\mathbf{Q}(t)$ linearly related to the displacement response vector $\mathbf{U}(t)$, i.e., $\mathbf{Q}(t) = \mathbf{B}\mathbf{U}(t)$, where $\mathbf{B}$ = constant matrix.

3.3 Response Statistics of MDOF Linear Systems Subjected to Modulated Coloured Noise

Time-modulated coloured noises constitute an important and widely used class of non-stationary dynamic load processes. The expression given in Eq. 20 describing a general non-stationary loading process reduces to

$$F(t) = A_F(t) \cdot P(t) \quad (30)$$

where the time-modulating function $A_F(t)$ is frequency-independent and the process $P(t)$ is a coloured noise with PSD function having the following general expression (i.e., rational function)

$$\Phi(\omega) = S_0 \sum_{k=1}^N \frac{A_k}{(\omega - \bar{\omega}_k)} \quad (31)$$

where A_k and $\bar{\omega}_k$ ($k = 1, 2, \dots, N$) = complex-valued constants and S_0 = real-valued scaling constant. In the sequel, it is assumed that the time-modulating functions have the following general expression

$$A_F(t) = H(t) \sum_{q=1}^M a_q e^{b_q t} \quad (32)$$

in which a_q and b_q ($q = 1, 2, \dots, M$) = real-valued constants, and $H(t)$ = unit-step function.

Substituting Eq. 32 into Eq. 22 yields ($i = 1, 2, \dots, 2n$)

$$A_{S_i}(\omega, t) = j \sum_{q=1}^M \left\{ a_q e^{b_q t} \cdot \left[\frac{e^{-j \cdot (\omega - \omega_{iq}) \cdot t} - 1}{\omega - \omega_{iq}} \right] \right\} \quad (33)$$

where $\omega_i = -j\lambda_i$ and $\omega_{iq} = \omega_i + jb_q$. The spectral characteristics defined in Eq. 29₁ can be computed using Cauchy's residue theorem as [21] ($i, m = 1, 2, \dots, 2n$)

$$\begin{aligned} C_{S_i S_m}(t) = S_0 \sum_{q=1}^M \sum_{s=1}^M \sum_{k=1}^N a_q a_s A_k e^{(b_q + b_s) \cdot t} \\ \cdot \left\{ \left[e^{j \cdot (\omega_{ms} - \omega_{iq}^*) \cdot t} + 1 \right] \cdot I_1^{iq, ms, k} \right. \\ \left. - e^{-j \omega_{iq}^* t} \cdot I_2^{iq, ms, k} - e^{j \omega_{ms} t} \cdot I_3^{iq, ms, k} \right\} \end{aligned} \quad (34)$$

in which

$$I_1^{iq, ms, k} = 2\pi j \sum_{r=1}^3 B_r^{iq, ms, k} \cdot I(\tilde{\omega}_r) \quad (35)$$

$$I_2^{iq, ms, k} = 2\pi j \sum_{r=1}^3 B_r^{iq, ms, k} \cdot e^{j \tilde{\omega}_r t} \cdot I(\tilde{\omega}_r) \quad (36)$$

$$I_3^{iq,ms,k} = -2\pi j \sum_{r=1}^3 B_r^{iq,ms,k} \cdot e^{-j\tilde{\omega}_r t} \cdot I(-\tilde{\omega}_r) \quad (37)$$

where $\tilde{\omega}_1 = \omega_{iq}^*$, $\tilde{\omega}_2 = \omega_{ms}$, $\tilde{\omega}_3 = \bar{\omega}_k$ and

$$B_r^{iq,ms,k} = \frac{1}{(\tilde{\omega}_r - \tilde{\omega}_p) \cdot (\tilde{\omega}_r - \tilde{\omega}_u)}, \quad r, p, u = 1, 2, 3 \quad r \neq p \neq u \quad (38)$$

while

$$I(\omega) = \max \left[\frac{\Im(\omega)}{|\Im(\omega)|}, 0 \right] \quad (39)$$

and $\Im(\cdot)$ = imaginary part of the quantity in parentheses. Madsen and Krenk [22], Krenk and Madsen [8] and Krenk et al. [23] applied the same approach (integration using Cauchy's residue theorem) to the real-valued (second-order) modal responses to derive the closed-form solutions for the auto- and cross-correlation functions of the response processes of classically damped *MDOF* linear systems subjected to white noise excitations modulated by rational time-modulating functions. After extensive algebraic manipulations [6], the spectral characteristics in Eq. 29₂ are obtained as ($i, m = 1, 2, \dots, 2n$)

$$\begin{aligned} C_{S_i \Sigma_m}(t) = & -jS_0 \sum_{q=1}^M \sum_{s=1}^M \sum_{k=1}^N a_q a_s A_k e^{(b_q + b_s)t} \\ & \cdot \left\{ \left[e^{j \cdot (\omega_{ms} - \omega_{iq}^*) \cdot t} + 1 \right] \cdot J_1^{iq,ms,k} \right. \\ & \left. - e^{-j\omega_{iq}^* t} \cdot J_2^{iq,ms,k} - e^{j\omega_{ms} t} \cdot J_3^{iq,ms,k} \right\} \quad (40) \end{aligned}$$

in which

$$J_1^{iq,ms,k} = - \sum_{r=1}^3 \left\{ B_r^{iq,ms,k} \cdot [\log(\tilde{\omega}_r) + \log(-\tilde{\omega}_r)] \right\} \quad (41)$$

$$J_2^{iq,ms,k} = 2 \sum_{r=1}^3 B_r^{iq,ms,k} \cdot \left\{ e^{j\tilde{\omega}_r t} \cdot \{ E_1(j\tilde{\omega}_r t) + j\pi \cdot I(\tilde{\omega}_r) \cdot \text{sign}[\Re(\tilde{\omega}_r)] \} \right\} \quad (42)$$

$$J_3^{iq,ms,k} = 2 \sum_{r=1}^3 B_r^{iq,ms,k} \cdot \left\{ e^{-j\tilde{\omega}_r t} \cdot \{ E_1(-j\tilde{\omega}_r t) - j\pi \cdot I(-\tilde{\omega}_r) \cdot \text{sign}[\Re(\tilde{\omega}_r)] \} \right\} \quad (43)$$

where $E_1(\cdot)$ denotes the integral exponential function defined as [9]

$$E_1(x) = \int_x^\infty \frac{e^{-u}}{u} du, \quad |\arg(x)| < \pi \quad (44)$$

and $\Re(\cdot)$ = real part of the quantity in parentheses.

It is noteworthy that the closed-form exact solutions presented here for the spectral characteristics of MDOF linear system response processes are valid for any kind of time-modulated coloured noise excitation as described by Eqs. 30–32. On the other hand, the considered description of these response processes includes only first and second order statistical moments and thus is complete only for Gaussian response processes, which are obtained only if the input process is a time-modulated Gaussian coloured noise. For the remainder of this chapter, only time-modulated Gaussian coloured noise excitations are considered.

4 Failure Probability Approximations for the First-Passage Reliability Problem

The first-passage problem is a classical random vibration problem which consists of computing the probability of a response quantity exceeding a deterministic threshold in a given interval of time when the parameters defining the structural system and/or the loading are random quantities. In this work, the structural system is considered as deterministic and the loading is modelled as a time-modulated coloured noise process. To date, no exact closed-form solution of this problem is available, even for the simplest case of structural model (deterministic linear elastic SDOF system) subjected to the simplest stochastic load model (stationary Gaussian white noise). The Monte Carlo simulation technique is the only general method accommodating for non-stationarity and non-Gaussianity of the excitation as well as nonlinearity in the structural behaviour and uncertainty/randomness in the structural parameters. However, it is computationally extremely expensive. Nevertheless, an analytical upper bound of the time-variant probability of failure can be obtained readily when response mean out-crossing rates are available [19] and several direct approximations of this failure probability have been developed making use of different statistics of the response quantities of interest [11, 24]. In particular, Poisson and Vanmarcke approximations have been shown to offer a good compromise between accuracy and computational effort [10, 16–18].

The mean upcrossing rates of the response of a linear MDOF system subjected to a time-modulated coloured noise process can be readily derived from the second-order statistics of the normalized complex modal responses. In fact, given that the input processes are Gaussian and the system (filter) is linear, the output processes are also Gaussian. Using the well-known Rice formula [16, 17], the mean upcrossing rate of the i th DOF displacement response process $U_i(t)$ relative to threshold ξ_i^+ , i.e., $\nu_{U_i}(\xi_i^+, t)$, can be obtained as

$$\nu_{U_i}(\xi_i^+, t) = \frac{\sigma_{\dot{U}_i} \sqrt{1 - \rho_{U_i \dot{U}_i}^2}}{2\pi\sigma_{U_i}} \cdot \exp\left(-\frac{r}{\rho_{U_i \dot{U}_i}^2}\right) \cdot [1 + \sqrt{\pi}r \cdot \exp(r^2) \cdot \operatorname{erfc}(-r)] \quad (45)$$

where

$$\rho_{U_i \dot{U}_i} = \frac{C_{U_i \dot{U}_i}(t)}{\sigma_{U_i}(t) \cdot \sigma_{\dot{U}_i}(t)} \quad (46)$$

$$r = \frac{\rho_{U_i \dot{U}_i}}{\sqrt{2 \cdot (1 - \rho_{U_i \dot{U}_i}^2)}} \cdot \left(\frac{\xi_i^+}{\sigma_{U_i}} \right) \quad (47)$$

and $erfc(\dots)$ denotes the error function [9]. The mean out-crossing rate of the level $|U_i| = \xi_i$, $v_{|U_i|}(\xi_i, t)$ (symmetric double-barrier problem [11]), is given by

$$v_{|U_i|}(\xi_i, t) = 2 \cdot v_{U_i}(\xi_i, t) \quad (48)$$

In the remainder of this study, only symmetric double-barrier problems are explicitly considered. For the sake of brevity and clarity, the subscripts indicating the considered processes are dropped. An upper bound of the probability of failure over the time interval $[0, t]$, $P_f(t)$, is obtained by integrating in time the mean out-crossing rate $v(\xi, t)$ of level ξ by the absolute value of the subject response quantity $U(t)$ [19], i.e.,

$$P_f(t) \leq E[N(t)] = \int_0^t v(\xi, \tau) \cdot d\tau \quad (49)$$

where $E[N(t)]$ denotes the expected number of out-crossing events in the time interval $[0, t]$. Moreover, it is common to express the probability of failure as [11, 18]

$$P_f(t) = 1 - P[|U(t = 0)| < \xi] \cdot e^{-\int_0^t h(\xi, \tau) \cdot d\tau} \quad (50)$$

where $P[|U(t = 0)| < \xi]$ = probability that at time $t = 0$ the absolute value of the response quantity $U(t)$ is below the failure threshold ξ , and $h(\xi, t)$ = hazard function, i.e., the mean out-crossing rate conditional to zero out-crossings prior to time t . In this work, at rest initial conditions are assumed, resulting in $P[|U(t = 0)| < \xi] = 1$.

The simplest approximation of the hazard function is the Poisson hazard function [16, 17], obtained by assuming that the out-crossing events follow the memoryless Poisson random occurrence model (i.e., out-crossing events are statistically independent). This simplifying assumption leads to

$$P_{f,P}(t) = 1 - e^{-\int_0^t v(\xi, \tau) \cdot d\tau} \quad (51)$$

The Poisson hazard function is asymptotically correct for high barrier/threshold levels and broadband processes, while it tends to give a very conservative estimate of the probability of failure for low thresholds and/or narrow-band processes.

Vanmarcke suggested two improved approximations [18], referred to herein as classical and modified Vanmarcke approximation, which both require the bandwidth parameter $q(t)$ of the stochastic response process $U(t)$ considered, in addition to the mean out-crossing rate of process $U(t)$. The classical and modified Vanmarcke approximations are expressed as

$$P_{f,VM}(t) = 1 - e^{-\int_0^t h_{VM}(\xi, \tau) \cdot d\tau} \quad (52)$$

$$P_{f,mVM}(t) = 1 - e^{-\int_0^t h_{mVM}(\xi, \tau) \cdot d\tau} \quad (53)$$

respectively, where

$$h_{VM}(\xi, t) = v(\xi, t) \cdot \frac{1 - e^{-\sqrt{\frac{\pi}{2}} \cdot q(t) \cdot \frac{\xi}{\sigma_U(t)}}}{1 - e^{-0.5\xi^2/\sigma_U^2(t)}} \quad (54)$$

$$h_{mVM}(\xi, t) = v(\xi, t) \cdot \frac{1 - e^{-\sqrt{\frac{\pi}{2}} \cdot [q(t)]^{1.2} \cdot \frac{\xi}{\sigma_U(t)}}}{1 - e^{-0.5\xi^2/\sigma_U^2(t)}} \quad (55)$$

Equations 54 and 55 provide the analytical expressions of the classical and modified Vanmarcke approximations of the hazard function (in short, classical and modified Vanmarcke hazard functions) corresponding to the double-barrier first-passage problem. Similar relations for the single-barrier problem can be found elsewhere [18]. In this work, the considered analytical approximations of the time-variant failure probability (i.e., $P_{f,P}$, $P_{f,VM}$, and $P_{f,mVM}$) are obtained by computing the integrals in Eqs. 51–53 by numerical quadrature.

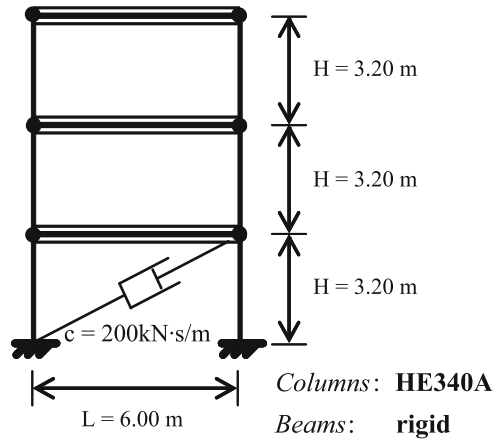
5 Benchmark Application

5.1 Benchmark Linear MDOF System: Three-Story Shear-Type Building

The three-story one-bay steel shear-frame shown in Fig. 1 is considered as an application example. This building structure has a uniform story height $H = 3.20$ m and a bay width $L = 6.00$ m. The steel columns are made of European *HE340A* wide flange beams with moment of inertia along the strong axis $I = 27,690.0$ cm⁴. The steel material is modelled as linear elastic with Young's modulus $E = 200$ GPa. The beams are considered rigid to enforce a typical shear building behaviour. Under this assumptions, the shear-frame is modelled as a 3 – DOF linear system.

The frame described above is assumed to be part of a building structure with a distance between frames $L' = 6.00$ m. The tributary mass per story, M , is obtained assuming a distributed gravity load of $q = 8$ kN/m², accounting for the structure's

Fig. 1 Geometric configuration of benchmark three-story one-bay shear-type steel frame



own weight, as well as for permanent and live loads, and is equal to $M = 28,800$ kg. The modal periods of the linear elastic undamped shear-frame are $T_1 = 0.38$ s, $T_2 = 0.13$ s and $T_3 = 0.09$ s, with corresponding effective modal mass ratios of 91.41%, 7.49% and 1.10%, respectively. Viscous damping in the form of Rayleigh damping is assumed with a damping ratio $\zeta = 0.02$ for the first and third modes of vibration. The same shear-frame is also considered with the addition of a viscous damper of coefficient $c = 200$ kN s/m across the first story as shown in Fig. 1. The structure with viscous damper is a non-classically damped system.

5.2 Earthquake Base Excitation

The benchmark structural model is subjected to a stochastic earthquake base excitation. The ground acceleration is modelled as a time-modulated coloured noise process. In this work, the Kanai-Tajimi spectrum [15] is used to model the basic stationary coloured noise process. The Kanai-Tajimi spectrum represents the stochastic process obtained filtering a white noise through a linear filter, representing the effects of a column of soil between the bedrock and the structure under consideration, i.e.

$$\Phi(\omega) = S_0 \cdot \left[\frac{\omega_g^4 + 4\zeta_g^2 \omega_g^2 \omega^2}{(\omega_g^2 - \omega^2)^2 + 4\zeta_g^2 \omega_g^2 \omega^2} \right] \quad (56)$$

where S_0 is the intensity of the ideal white noise, ω_g is the filter frequency that determines the dominant input frequency, and ζ_g is the filter damping coefficient that indicates the sharpness of the *PSD* function. This study assumes $\omega_g = 12.5$ rad/s, $\zeta_g = 0.6$, and $S_0 = 200$ cm²/s³ (Fig. 2). It can be easily shown that the Kanai-Tajimi PSD represents a special case of Eq. 31.

Fig. 2 One-sided Kanai-Tajimi PSD

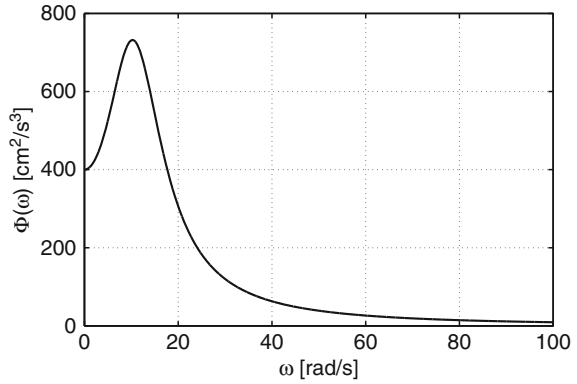
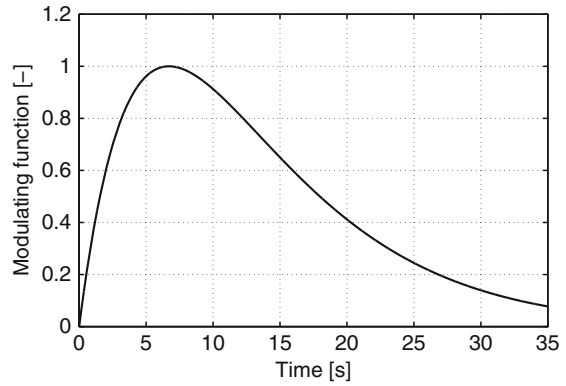


Fig. 3 Shinozuka and Sato's modulating function



The input process is then made non-stationary in amplitude by using the well-known modulating function of Shinozuka and Sato [25], defined as

$$A_F(t) = C \cdot \left[e^{-B_1 t} - e^{-B_2 t} \right] \cdot H(t) \quad (57)$$

where

$$C = \frac{B_1}{B_2 - B_1} \cdot e^{\frac{B_2}{B_2 - B_1} \cdot \log\left(\frac{B_2}{B_1}\right)} \quad (58)$$

and $B_2 > B_1 > 0$. By changing the values of parameters B_1 and $B_2 > 0$, a wide range of different time-modulating functions can be obtained. The modulating function of Shinozuka and Sato has been used extensively in random vibration studies regarding computation of probability density distributions of *SDOF* oscillators subjected to non-stationary excitation [26–28]. In this study, the following values for the modulating function parameters have been assumed: $B_1 = 0.045\pi$ and $B_2 = 0.05\pi$ (see Fig. 3). The Shinozuka and Sato's modulating function (Eq. 57) is a special case of Eq. 32.

The obtained input ground motion is a stochastic process which is stationary in frequency content but non-stationary in amplitude. It is noteworthy that the presented closed-form solutions for the spectral characteristics of *MDOF* linear system response processes are valid for input processes more general than the one considered here. In particular, fully non-stationary (i.e., both in amplitude and frequency content) processes can be modelled by superposing different coloured noise processes, each modulated by different time-modulating functions.

5.3 Spectral Characteristics of the Response Processes for the Benchmark Structure

Figures 4–7 show the time histories of (1) the variances of the floor displacements relative to ground (in short, relative displacements), (2) variances of the floor velocities relative to ground (in short, relative velocities), (3) bandwidth parameters, and (4) central frequencies (normalised by the first mode natural frequency) of the floor relative displacement responses. All quantities are provided for both the classically (without damper: CD) and non-classically damped (with damper: NCD) case. All presented closed-form solutions have been verified by Monte Carlo simulation.

Figures 4 and 5 show that in the NCD case, compared to the CD case, (1) the peak values of both relative displacement and velocity variances reduce significantly, and (2) the peak values of these variances are reached earlier. Moreover, the variance time histories have a shape very similar to the shape of the time-modulating function. It can be concluded that the introduction of the viscous damping device is a very effective retrofit option for the given structural systems, as shown by the decreased values of the variances of both floor relative displacements and velocities.

Figure 6 shows that the bandwidth parameters for the floor relative displacement response processes decrease rapidly from a value close to one (corresponding to a broadband process) to a value smaller than 0.2 (corresponding to a much

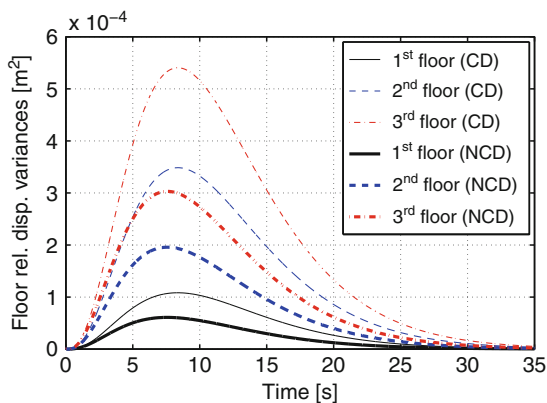


Fig. 4 Time-variant time histories of the floor relative displacement variances for both classically (without viscous damper: CD) and non-classically (with viscous damper: NCD) damped structures

Fig. 5 Time-variant time histories of the floor relative velocity variances for both classically (without viscous damper: CD) and non-classically (with viscous damper: NCD) damped structures

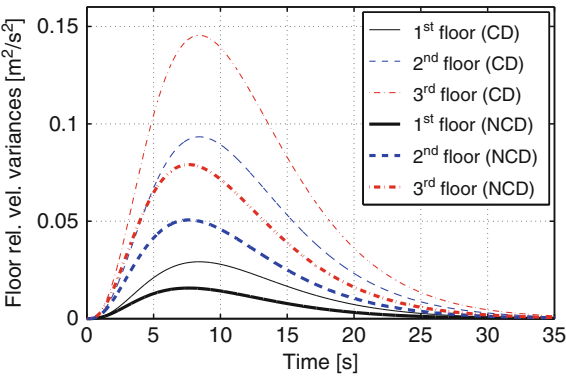


Fig. 6 Time-variant bandwidth parameter of the floor relative displacement processes for both classically (without viscous damper: CD) and non-classically (with viscous damper: NCD) damped structures

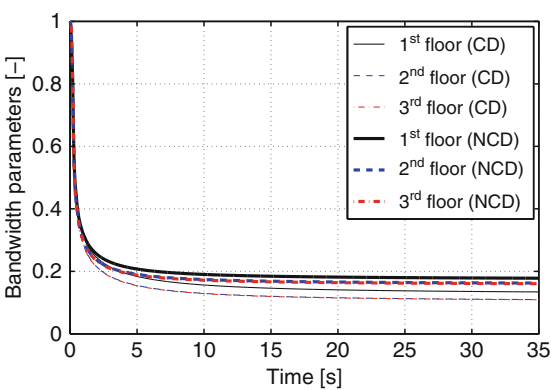
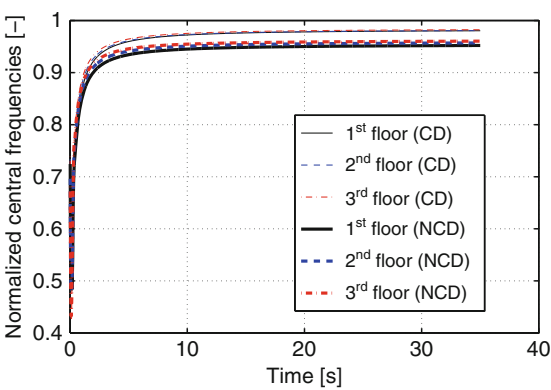


Fig. 7 Time-variant central frequency (normalised by the first mode natural circular frequency) of the floor relative displacement processes for both classically (without viscous damper: CD) and non-classically (with viscous damper: NCD) damped structures



narrower band process). In addition, Fig. 6 shows that, for the CD case, the time histories of time-variant bandwidth parameters for the second and third floor relative displacement response processes are almost coincident, while they differ by a non-negligible amount from the bandwidth parameter time history of the first floor

relative displacement. In particular, the asymptotic value of the bandwidth parameters for second and third floor relative displacement processes are smaller than the corresponding quantity for the first floor relative displacement response, i.e., the first floor relative displacement response is asymptotically a narrower band process than second and third floor relative displacement responses [5]. For the NCD case, the differences among the bandwidth parameter time histories among first, second and third floor relative displacement processes are smaller than for the CD case. The asymptotic values of the floor relative displacement bandwidth parameters are larger for the NCD case than for the CD case, i.e., the floor relative displacement response processes are broader in band for the NCD structure than for the CD one.

Figure 7 shows that the normalised central frequency time histories reach in a very short time (less than 5 s) an asymptotic value. In Fig. 7, it is also observed that the normalised central frequency asymptotic values are all only slightly lower than one, i.e., the central frequency asymptotic values for all floor relative displacement response processes are very close to the natural frequency of the structural system. This phenomenon indicates that floor relative displacement response processes are dominated by the first mode of vibration of the structure, as expected from the values of the modal mass ratios obtained from the modal analysis of the undamped system. In addition, the central frequency asymptotic values of the floor relative displacement response processes are smaller for the NCD case than for the CD one. This fact suggests that, in the NCD structure, higher modes have a larger influence on the floor relative displacement response processes.

5.4 Failure Probability Estimates for the Benchmark Structure

The closed-form exact solutions for the spectral characteristics of the floor relative displacement and velocity responses of the benchmark structure are used here to analytically estimate the probability that the interstorey drifts out-cross given deterministic thresholds. The threshold $\xi = 0.032$ m (corresponding to an interstorey drift ratio equal to 0.01) is considered here. This value is chosen for two reasons: (1) for higher interstorey drift ratios, the assumption of structural linear elastic behaviour is not realistic any more, and (2) the corresponding failure probabilities are sufficiently large to allow the use of Monte Carlo simulation (MCS) to validate the approximate analytical estimates derived in this work. For both the CD and NCD structure, 10,000 realisations of the response time histories for all three floor relative displacements and velocities are generated. The earthquake ground motion inputs are simulated by using the Spectral Decomposition Method [25] and the response time history were obtained through the exact piece-wise linear integration algorithm [29].

Figure 8 compares three different analytical approximations of the hazard function for the first interstorey drift of the CD structure subjected to the previously described earthquake base excitation and relative to the threshold $\xi = 0.032$ m for the symmetric double-barrier problem. The three considered approximations are: (1) the Poisson hazard function (i.e., the mean out-crossing rate), (2) the classical

Fig. 8 Approximate hazard functions for the first interstory drift of the classically damped structure

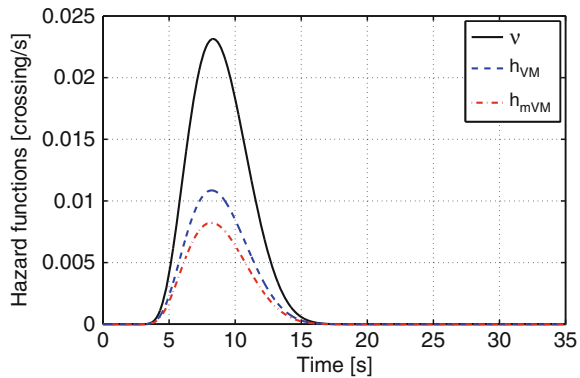
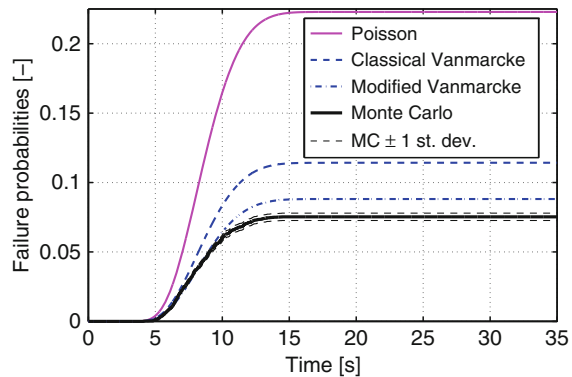


Fig. 9 Failure probability estimates for the first interstory drift of the classically damped structure



Vanmarcke hazard function, and (3) the modified Vanmarcke hazard function. It is observed that the two Vanmarcke hazard functions assume significantly smaller values than the mean out-crossing rate function. This difference was expected, since the chosen threshold level is not very high (compared to the maximum value of the response standard deviation during the entire time history analysis) for the considered structure and, consequently, the corresponding out-crossings at different time instants can be correlated, leading to an overestimation of the actual hazard function.

Figure 9 compares three different analytical approximations of the time-variant failure probability for the first interstory drift of the CD structure, namely (1) the Poisson approximation, (2) the classical Vanmarcke approximation, and (3) the modified Vanmarcke approximation. Figure 9 plots also the MCS estimate of the time-variant failure probability and the corresponding confidence interval of $\pm$ one standard deviation of the failure probability estimator. The results are provided for 35 s of analysis.

In Fig. 9, it is observed that, as expected, the Poisson approximation largely overestimates (about three times) the MCS-based failure probability. On the other hand, the two Vanmarcke approximations and particularly the modified Vanmarcke approximation provide a substantially improved estimate of the failure probability, even though they still overestimate (with errors equal to 51% and 17%, respectively) the failure probability after 35 s of analysis.

Fig. 10 Approximate hazard functions for the first interstory drift of the non-classically damped structure

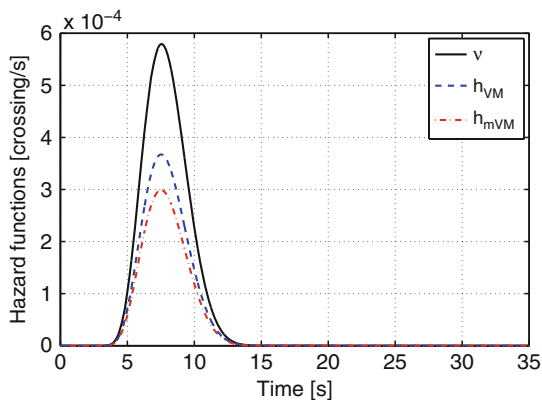
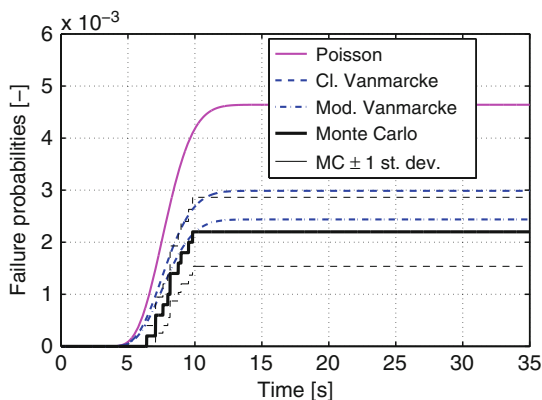


Fig. 11 Failure probability estimates for the first interstory drift of the non-classically damped structure



To evaluate probabilistically the effectiveness of using the viscous damper device in reducing the failure probability of the considered structure, a time-variant reliability analysis is repeated for the first interstory drift of the NCD structure considering the same displacement threshold $\xi = 0.032$ m and the same stochastic input earthquake ground motion. The results of this analysis are presented in Figs. 10 and 11, similarly to the results presented in Figs. 8 and 9 for the CD structure. In particular, Fig. 10 plots the time histories from 0 to 35 s of the three considered analytical approximations of the hazard function, and Fig. 11 compares the time histories from 0 to 35 s of the three considered analytical approximations of the time-variant failure probability with the corresponding MCS results.

The differences between Poisson hazard function and the two Vanmarcke hazard functions shown in Fig. 10 for the NCD case are smaller than for the CD case (see Fig. 8) but still non-negligible. In Fig. 11, it is observed that the Poisson approximation significantly overestimates the MCS failure probability estimate (with error equal to 110% at time $t = 35$ s). On the other hand, the classical and modified

Vanmarcke approximations provide failure probability estimates which are 36% and 11% larger, respectively, than the MCS result at time $t = 35$ s. It is noteworthy that, for the NCD structure, the coefficient of variation of the MCS failure probability estimator is much larger than for the classically damped case, as shown by the large confidence interval of the MCS estimate in Fig. 11. It is concluded that, also in this case, the two Vanmarcke approximations provide more accurate estimates of the failure probability when compared to the simpler Poisson approximation.

6 Conclusions

In this work, the spectral characteristics of non-stationary random processes are applied to the time-variant first-passage problem in structural reliability. In particular, new closed-form exact solutions are presented for the non-geometric spectral characteristics (*NGSCs*) of the non-stationary random responses of linear elastic classically and non-classically damped multi-degree-of-freedom structures subjected to earthquake base excitation modelled as a filtered white noise process defined by the Kanai-Tajimi power spectral density and time-modulated by the Shinozuka and Sato modulating function. The *NGSCs* are essential for computing the time-variant bandwidth parameter and central frequency of non-stationary response processes of linear systems. The time-variant bandwidth parameter is used to compute analytically the classical and modified Vanmarcke approximations of the first-passage problem failure probability, which are expressed as integrals in time of the closed-forms of the corresponding hazard functions. These approximate solutions to the first-passage problem are compared with the classical Poisson approximation and Monte Carlo simulation results for an idealised linear elastic model of a three-story one-bay shear-type steel building. The retrofit of this benchmark structure with viscous dampers is also considered, allowing (1) to illustrate the use of the newly available closed-form approximations of the failure probability for non-classically damped linear elastic systems, and (2) to show an example of practical use in structural engineering of the presented analytical derivations. The failure condition is expressed in terms of interstory drifts out-crossing specified deterministic thresholds. The results presented in this study show that the two Vanmarcke approximations can improve considerably the estimates of the failure probability for the first-passage problem when compared with the simpler Poisson approximation.

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References

1. Priestley, M.B.: Spectral Analysis and Time Series, vol. 1: Univariate Series, vol. 2: Multivariate Series, Prediction and Control. Academic, London, Fifth Printing (1987)
2. Nigam, N.C.: Introduction to Random Vibrations. MIT Press, Cambridge (1983)
3. Di Paola, M.: Transient spectral moments of linear systems. *SM Arch.* **10**, 225–243 (1985)
4. Michaelov, G., Sarkani, S., Lutes, L.D.: Spectral characteristics of nonstationary random processes – a critical review. *Struct. Saf.* **21**(3), 223–244 (1999)
5. Barbato, M., Conte, J.P.: Spectral characteristics of non-stationary random processes: Theory and applications to linear structural models. *Probabilistic Eng. Mech.* **23**(4), 416–426 (2008)
6. Barbato, M., Vasta, M.: Closed-form solutions for the time-variant spectral characteristics of non-stationary random processes. *Probabilistic Eng. Mech.* **25**(1), 9–17 (2010)
7. Michaelov, G., Sarkani, S., Lutes, L.D.: Spectral characteristics of nonstationary random processes – response of a simple oscillator. *Struct. Saf.* **21**(3), 245–267 (1999)
8. Krenk, S., Madsen, P.H.: Stochastic response analysis. In: Thoft-Christensen, P., Nijhoff, M. (ed.) NATO ASI Series: Reliability Theory and Its Application in Structural and Soil Mechanics, pp. 103–172 (1983)
9. Abramowitz, M., Stegun, I.A.: Exponential integral and related functions, Ch. 5 in Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing Dover, New York, pp. 227–233 (1972)
10. Corotis, R.B., Vanmarcke, E.H., Cornell, C.A.: First passage of nonstationary random processes. *J. Eng. Mech. Div. ASCE*, **98**(EM2), 401–414 (1972)
11. Crandall, S.H.: First-crossing probabilities of the linear oscillator. *J. Sound Vib.* **12**(3), 285–299 (1970)
12. Barbato, M., Conte, J.P.: Extension of spectral characteristics to complex-valued random processes and applications in structural reliability. ICASP10, Tokyo, Japan, August 1–3 (2007)
13. Arens, R.: Complex processes for envelopes of normal noise. *IRE Trans. Inf. Theory* **3**, 204–207 (1957)
14. Dugundji, J.: Envelope and pre-envelope of real waveforms. *IRE Trans. Inf. Theory* **4**, 53–57 (1958)
15. Clough, R.W., Penzien, J.: Dynamics of Structures. McGraw-Hill, 2nd Ed., New York (1993)
16. Rice, S.O.: Mathematical analysis of random noise. *Bell Syst. Tech. J.* **23**, 282–332 (1944)
17. Rice, S.O.: Mathematical analysis of random noise. *Bell Syst. Tech. J.* **24**, 146–156 (1945)
18. Vanmarcke, E.H.: On the distribution of the first-passage time for normal stationary random processes. *J. Appl. Mech. ASME* **42**(1), 215–220 (1975)
19. Lin, Y.K.: Probabilistic Theory of Structural Dynamics. McGraw-Hill, New York (1967); Krieger, Huntington (1976)
20. Reid, J.G.: Linear System Fundamentals: Continuous and Discrete, Classic and Modern. McGraw-Hill, New York (1983)
21. Barbato, M., Conte, J.P.: Spectral characteristics of non-stationary stochastic processes: theory and applications to linear structural systems. Report SSR-07-23, University of California at San Diego, La Jolla (2007)
22. Madsen, P.H., Krenk, S.: Stationary and transient response statistics. *J. Eng. Mech. Div. ASCE* **108**(EM4), 622–634 (1982)
23. Krenk, S., Madsen, H.O., Madsen, P.H.: Stationary and transient response envelopes. *J. Eng. Mec. ASCE* **109**(1), 263–278 (1983)
24. Wen, Y.K.: Approximate methods for nonlinear time-variant reliability analysis. *J. Eng. Mec. ASCE* **113**(12), 1826–1839 (1987)
25. Shinozuka, M., Sato, Y.: Simulation of nonstationary random processes. *J. Eng. Mech. Div. ASCE* **93**(EM1), 11–40 (1967)
26. Solomos, G.P., Spanos, P.T.D.: Structural reliability under evolutionary seismic excitation. *Soil Dyn. Earthquake Eng.* **2**(2), 110–116 (1983)

27. Spanos, P.T.D., Solomos, G.P.: Barrier crossing due to transient excitation. *J. Eng. Mech. ASCE* **110**(1), 20–36 (1984)
28. Spanos, P.T.D., Solomos, G.P.: Oscillator response to nonstationary excitation. *J. Appl. Mech. ASME* **51**(4), 907–912 (1984)
29. Chopra, A.K.: *Dynamics of Structures: Theory and Applications to Earthquake Engineering*. 2nd edn., Prentice Hall, Upper Saddle River (2001)

Stochastic Seismic Analysis of Large Linear Structural Systems Under Fully Non-stationary Spectrum Compatible Ground Motion

Pierfrancesco Cacciola and Giuseppe Muscolino

Abstract Seismic assessment of linear systems is usually performed adopting the well-known modal analysis along with response spectrum technique. Limits of this approach are directly related to the hypothesis of stationary behavior of the response adopted for deriving most common modal combination rules, i.e. SRSS, CQC. Generally, the response of a structure under seismic actions is non-stationary and pertinent analyses have to be performed. In this chapter stationary and non-stationary stochastic models of the seismic action consistent with a given response spectrum will be initially discussed. Furthermore, a technique for determining the non-stationary response through the so-called non-geometric spectral moments will be addressed. Finally a modal correction technique will be proposed in order to make the proposed procedure competitive for coping with the analysis of large structural systems vibrating under random base excitations.

1 Introduction

To date, elastic response spectrum is the most common method used by the practitioners to represent the seismic action according to various international codes. Limits of response spectrum technique are well recognized. In order to overcome these drawbacks seismic codes [15, 19] proposed to use alternative approaches for design applications such as the use of “appropriate simulated” time-histories. Interestingly, also in the simplest case of classically damped linear behaving structures, response spectrum techniques possess various drawbacks. Specifically, the approximations adopted to derive the well-known complete quadratic modal combination

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(CQC) rule on the nodal response parameter leads to noticeable errors in the high frequency range, due to the approximation on the correlation coefficients [3, 10, 17]. Furthermore, due to the assumption of equal modal and nodal peak factors CQC leads to large errors also in the case of slender structures having well spaced modal frequencies [3]. A significant improvement of the accuracy on the seismic assessment can be achieved by removing the approximations introduced for deriving the combination rules required by the response spectrum techniques leading to the Stationary Stochastic Seismic Analysis. Moreover, the seismic action is non-stationary in nature and further limits in the response spectrum technique are directly connected with the hypothesis of stationary response retained in the derivation of the traditional modal combination rules. The latter limits can be overcome only by removing the hypothesis of stationary behavior of the response. Accordingly, a fully Non-stationary Stochastic Seismic analysis has to be performed.

In this chapter the non-stationary seismic stochastic analysis of linear behaving structures is addressed. To this aim the spectrum compatible evolutionary power spectral density function of ground motion is first evaluated, according to model recently proposed by Cacciola [2]. Furthermore, a new computationally competitive method for the seismic stochastic analysis of large linear behaving system vibrating under fully non-stationary Gaussian excitations is proposed. The method is based on the use of modal analysis, in the evaluation of stochastic response, along with a modal correction method, which includes the contribution of neglected modes in approximate form. In particular, an extension of the so-called mode-acceleration method (MAM), originally proposed in literature for deterministic input [21], to the evaluation of the frequency domain non-stationary stochastic response is presented.

Numerical results from the study of a large structural system show the accuracy and the efficiency of the proposed technique for determining the non-geometric response spectral moments through a modal approach. Furthermore, since the mode-acceleration method is able to correct only the displacement statistics, as shown in [4], the efficiency of this approach for determining high order non-geometric response spectral moments will be also discussed.

2 Spectrum Compatible Ground Motion Models

In the framework of seismic engineering it is well known that only a probabilistic approach can afford a reliable representation of earthquake ground motion. Indeed seismic waves are affected by several factors, i.e. definition of source, path, site effects, etc., that generally cannot be defined in a deterministic fashion. As a consequence the modeling of ground motion acceleration, $\ddot{u}_g(t)$, as a zero mean Gaussian stochastic process is the model more commonly used by the scientific community. In this model the seismic input is fully defined by the knowledge of its autocorrelation function $R_{\ddot{u}_g}(t_1, t_2)$ or by the so-called evolutionary power spectral density function $S_{\ddot{u}_g}(\omega, t)$. Note that, in the seismic stochastic analysis the one-sided power

spectral density function is often introduced; the latter can be suitably defined, in the Priestley representation [29] by the following equation

$$\begin{aligned} G_{\ddot{u}_g}(\omega, t) &= 2S_{\ddot{u}_g}(\omega, t) = |a(\omega, t)|^2 G(\omega), \quad \omega \geq 0; \\ G_{\ddot{u}_g}(\omega, t) &= 0, \quad \omega < 0. \end{aligned} \quad (1)$$

where $a(\omega, t)$ is the frequency dependent modulating complex function while $G(\omega)$ is the one-sided power spectral density function of the stationary counterpart of $\ddot{u}_g(t)$. It follows that the autocorrelation function can be defined as

$$R_{\ddot{u}_g}(t_1, t_2) = \int_0^\infty \exp[i\omega(t_2 - t_1)] a(\omega, t_1) a^*(\omega, t_2) G(\omega) d\omega \quad (2)$$

where the asterisk means complex conjugate quantity and $i = \sqrt{-1}$ is the imaginary unit. Note that since the one-sided power spectral density is introduced, the integral (2) leads to a complex autocorrelation function [12]. Moreover, in earthquake engineering applications the function $a(\omega, t)$ only makes sense, in frequency domain, if it is a non-negative real and symmetric function [36].

In absence of specific study on local geological and seismological condition, for design application a valuable strategy for modelling the seismic action is to relate ground motion random processes to the response spectra provided by the seismic codes, i.e. through the definition of proper spectrum compatible models.

2.1 Quasi-Stationary Model

According to the theory of stochastic analysis, by assuming the ground-acceleration process as zero-mean Gaussian stationary process, the pseudo-acceleration response spectrum, $S_{pa}(\omega_0, \xi_0)$, for a given damping ratio ξ_0 and natural circular frequency ω_0 , can be related to the mean value of largest peak of the response by means of the following expression [8, 9, 34]

$$S_{pa}(\omega_0, \xi_0) = \omega_0^2 \eta_U(T_s; p = 0.5; \lambda_{0,U}, \lambda_{1,U}, \lambda_{2,U}) \sqrt{\lambda_{0,U}}, \quad (3)$$

where η_U is the peak factor, T_s is the time observing window, p is the not-exceeding probability and $\lambda_{i,U}$, $i = 0, 1, 2$, are the response spectral moments defined as

$$\lambda_{i,U} = \int_0^\infty \omega^i |H(\omega)|^2 G(\omega) d\omega, \quad (4)$$

in which $|H(\omega)|^2 = ((\omega_0^2 - \omega^2)^2 + 4\xi_0^2 \omega_0^2 \omega^2)^{-1}$ is the energy transfer function of the oscillator. It is noted also that in Eq. 3 $p = 0.5$ defines the 50% fractile of the response. It is usually assumed that it is very near to the mean value of the largest peak of the oscillator response.

It is useful to observe that Eq. 3 affords a vehicle for determining the unknown power spectral density of the stationary counterpart of the ground acceleration process $G(\omega)$ (see e.g. [11, 13, 16, 27, 34]). A handy recursive expression determining the power spectral density compatible with a given response spectrum has been proposed by Cacciola et al. [3]. Specifically,

$$G(\omega_i) = \begin{cases} 0, & 0 \leq \omega \leq \omega_\alpha \\ \frac{4\xi_0}{\omega_i \pi - 4\xi_0 \omega_{i-1}} \left(\frac{S_{pa}^2(\omega_i, \xi_0)}{\bar{\eta}_U^2(\omega_i, \xi_0)} - \Delta\omega \sum_{k=1}^{i-1} G(\omega_k) \right), & \omega > \omega_\alpha \end{cases} \quad (5)$$

where $\bar{\eta}_U$ is the peak factor approximately determined according to the hypothesis of a barrier out-crossing in clumps and spectral moments determined assuming that the input PSD possesses a smooth shape and $\xi_0 \ll 1$:

$$\bar{\eta}_U(\omega_i, \xi_0) = \sqrt{2 \ln \left\{ 2N_U \left[1 - \exp \left[-\delta_U^{1,2} \sqrt{\pi \ln(2N_U)} \right] \right] \right\}}, \quad (6)$$

with

$$N_U = \frac{T_s}{2\pi} \omega_i (-\ln p)^{-1}, \quad \delta_U = \left[1 - \frac{1}{1 - \xi_0^2} \left(1 - \frac{2}{\pi} \arctan \frac{\xi_0}{\sqrt{1 - \xi_0^2}} \right)^2 \right]^{1/2}. \quad (7)$$

Moreover, in Eq. 5, $\omega_\alpha \cong 1$ rad/s is the lowest bound of the existence domain of $\bar{\eta}_U$.

The accuracy of Eq. 5 for determining the spectrum-compatible power spectral density is generally satisfactory; however accuracy can be improved applying iteratively the equation

$$G^{(1)}(\omega) = G(\omega); \quad G^{(j)}(\omega) = G^{(j-1)}(\omega) \left[\frac{S_{pa}^2(\omega, \xi_0)}{S_{pa}^{2(j-1)}(\omega, \xi_0)} \right], \quad (8)$$

$S_{pa}^{2(j)}$ being the approximate pseudo-acceleration spectrum determined at the j -th iteration.

Once the spectrum-compatible power spectral density $G(\omega)$ is evaluated, it is possible to simulate the r -th sample of stationary or quasi-stationary spectrum-compatible ground acceleration via the superposition of N_a harmonics with random phases [30]. That is

$$\ddot{u}_g^{(r)}(t) = a(t) \sum_{i=1}^{N_a} \sqrt{2G_{\ddot{u}_g}(i \Delta\omega) \Delta\omega} \cos(i \Delta\omega t + \varphi_i^{(r)}) \quad (9)$$

$\varphi_i^{(r)}$ are independent random phases uniformly distributed in the interval $[0, 2\pi)$ and $a(t)$ is the modulating function, which for stationary samples is assumed equal to

one. Note that the introduction of a modulating function different from one will make the sample generated by Eq. 9 as quasi-stationary. As a consequence the results from the Monte Carlo Simulation and from the Stationary Stochastic Seismic Analysis will be in general different. In order to reduce the differences it is necessary that the response is stationary within a time window of duration T_s . To achieve this objective the simplest way is to select among the several modulating functions proposed in literature a modulating function with a constant time segment of duration T_s . Using this approach the spectrum-compatible criteria provided by seismic codes are satisfied [3]. Then the function

$$G_{u_g}^S(\omega, t) = a^2(t)G(\omega) \quad (10)$$

represents the evolutionary separable spectrum compatible power spectral density function.

2.2 Fully Non-stationary Model

To date spectrum-compatible criteria provided by seismic codes refers mainly to response characteristics. No rules have been imposed regarding the duration, the input energy, the stationary or not-stationary behaviour pertinent to the simulated time-history. On the other hand, the importance of non-stationary frequency content on the response of nonlinear structures has been manifested in various studies (see e.g. [35, 37]). Thus, more reliable simulations have to take into account the time variability of the frequency content of the ground motion. Consequently various procedures have been proposed in literature for determining non-stationary spectrum-compatible earthquakes.

Based on a statistical approach, Spanos and Vargas Loli [32] proposed a method for generating non-stationary spectrum-compatible earthquakes. The method is based on a preliminary estimation of a reliable evolutionary power spectral density of the ground motion and correcting the single simulated time history via an iterative scheme operating in the frequency domain. Generation of non-separable artificial earthquake accelerograms has been also proposed by Preumont [28]. The method assumes an empirical model of the evolutionary power spectral density function possessing the feature that high frequency component are magnified in the early part of the process and the iterative correction of the simulated accelerograms. Remarkably, the above described contributions focus mainly in the simulation of nonstationary artificial accelerograms and do not provide the spectrum compatible evolutionary power spectral density function useful for the direct stochastic seismic analysis. To overcome this drawback, Cacciola [2] recently proposed a model that allows the simulation of fully nonstationary accelerograms compatible with a given response spectrum as well as the straightforward evaluation of the pertinent non-separable power spectral density function. Accordingly, it is assumed that the non-stationary spectrum-compatible earthquake is modeled by the superposition of

two independent contributions: the first one is a fully non-stationary known counterpart modeled by a recorded or alternatively by a simulated earthquake, that takes into account the time variability of both intensity and frequency content; the second one is a corrective term represented by a quasi-stationary zero-mean Gaussian process that adjusts the response spectrum of the non-stationary signal in order to make it spectrum-compatible. That is:

$$\ddot{u}_g(t) = \ddot{u}_g^R(t) + \ddot{u}_g^S(t), \quad (11)$$

where $\ddot{u}_g^R(t)$ is the non-stationary signal that is assumed known and $\ddot{u}_g^S(t)$ is the random spectrum compatible corrective term whose power spectral density has to be determined. Taking into account the statistical independence of the two contributions the evolutionary spectrum-compatible power spectral density function is given by the superposition of two terms:

$$G_{\ddot{u}_g}(\omega, t) = G_{\ddot{u}_g}^R(\omega, t) + G_{\ddot{u}_g}^S(\omega, t) \quad (12)$$

where $G_{\ddot{u}_g}^R(\omega, t)$ is the joint time–frequency distribution of the recorded accelerogram [6, 7], while $G_{\ddot{u}_g}^S(\omega, t)$ is the separable power spectral density defining the corrective term. By using the fully non-stationary analytical ground motion model proposed by Conte and Peng [7] the first term of Eq. 12, $G_{\ddot{u}_g}^R(\omega, t)$, is assumed to be given by a superposition of a number of zero-mean, independent, uniformly modulated Gaussian processes given by the following equation

$$G_{\ddot{u}_g}^R(\omega, t) = \sum_{j=1}^N |a_j(t)|^2 G_j(\omega) \quad (13)$$

where

$$a_j(t) = \alpha_j(t - \vartheta_j)^{\beta_j} \exp(-\gamma_j(t - \vartheta_j)) U(t - \vartheta_j) \quad j = 1, \dots, N \quad (14)$$

$U(t - \vartheta_j)$ being the unit step function and

$$G_j(\omega) = \frac{\nu_j}{\pi} \left[\frac{1}{\nu_j^2 + (\omega + \eta_j)^2} + \frac{1}{\nu_j^2 + (\omega - \eta_j)^2} \right] \quad j = 1, \dots, N \quad (15)$$

Note that the parameters $N, \alpha_j, \beta_j, \gamma_j, \vartheta_j, \eta_j, \nu_j$ are determined by a best fit procedure in order to minimize the differences in a least square sense between the analytical model and the joint time–frequency distribution of a selected earthquake.

Furthermore, the second term of Eq. 12, $G_{\ddot{u}_g}^S(\omega, t)$, is given by

$$G_{\ddot{u}_g}^S(\omega, t) = a_0^2(t) G_0(\omega) \quad (16)$$

where, according to the quasi-stationary model described by Eqs. 9 and 10, $a_0(t)$ is the modulating function and [2]

$$G_0(\omega_i) = \begin{cases} 0, & 0 \leq \omega \leq \omega_\alpha \\ \frac{4\xi_0}{\omega_i \pi - 4\xi_0 \omega_{i-1}} \left(\frac{S_{pa}^2(\omega_i, \xi_0) - (S_{pa}^R(\omega_i, \xi_0))^2}{\bar{\eta}_U^2(\omega_i, \xi_0)} - \Delta\omega \sum_{k=1}^{i-1} G_0(\omega_k) \right), & \omega > \omega_\alpha \end{cases} \quad (17)$$

where $\bar{\eta}_U$ is the peak factor defined in Eq. 6 and S_{pa}^R is the pseudo-acceleration response spectrum for the recorded ground motions $\ddot{u}_g^R(t)$. It is noted that the iterative scheme is given by

$$G_0^{(1)}(\omega) = G_0(\omega); \quad G_0^{(j)}(\omega) = G_0^{(j-1)}(\omega) \left[\frac{S_{pa}^2(\omega, \xi_0)}{S_{pa}^{(j-1)}(\omega, \xi_0)^2} \right], \quad (18)$$

where $S_{pa}^{(j)}$ represent the response spectrum, of the stochastic ground motion model defined in Eq. 11, determined, at the j -th iteration. It is noted that Eq. 18 is defined for $S_{pa}(\omega_i, \xi_0) > S_{pa}^R(\omega_i, \xi_0)$. As a consequence a preliminary scaling procedure is usually required. Furthermore it is noted that from a physical point of view the power spectral density of the corrective term $G_{\ddot{u}_g}^S(\omega, t)$ represents the missing energy that an individual accelerogram requires for matching the target response spectrum.

After determining the non separable spectrum-compatible power spectral density $G_{\ddot{u}_g}^S(\omega, t)$ the simulation of samples of fully non-stationary spectrum-compatible ground acceleration processes is pursued via the equation

$$\ddot{u}_g^{(r)} = \ddot{u}_g^R(t) + a_0(t) \sum_{i=1}^{N_a} \sqrt{2G_0(i\Delta\omega)\Delta\omega} \cos(i\Delta\omega t + \varphi_i^{(r)}), \quad (19)$$

Remarkably, the non-stationary behavior of the spectrum compatible model relies on the recorded signal that would be chosen so to reflect local geotechnical and seismological characteristics. The spectrum-compatible criteria are satisfied by the superposition of the corrective quasi-stationary Gaussian random process whose power spectral density has been determined in order to scale the recorded spectrum. It is noted that due to the mathematical model proposed by Conte and Peng [7] assuming $Q_j(t)$ real and non-negative according to [36], Eq. 12 can be rewritten in the form

$$\begin{aligned} G_{\ddot{u}_g}(\omega, t) &= G_{\ddot{u}_g}^R(\omega, t) + G_{\ddot{u}_g}^S(\omega, t) = \sum_{j=1}^N a_j(t)^2 G_j(\omega) + a_0^2(t) G_0(\omega) \\ &= \sum_{j=0}^N a_j(t)^2 G_j(\omega) \end{aligned} \quad (20)$$

affording a compact representation of the evolutionary spectrum-compatible power spectral density function.

3 Deterministic Seismic Analysis

Once a reliable ground motion model is defined, the seismic analysis can be performed. In this section basic concepts of deterministic seismic analysis of linear behaving structures are briefly recalled. This makes possible to evidence the extension of the proposed approach to the case of stochastic excitation.

Let consider the equation of motion of a linear quiescent n -degree-of-freedom (n -DOF) structure subjected to seismic excitation, represented by the acceleration $\ddot{u}_g(t)$, written in the form

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = -\mathbf{M}\boldsymbol{\tau}\ddot{u}_g(t) \quad (21)$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the $n \times n$ mass, damping, and stiffness matrices of the structure; $\mathbf{u}(t)$ is the vector of displacements relative to the ground; $\boldsymbol{\tau}$ is the n order influence vector; and a dot over a variable denotes differentiation with respect to time.

Under the assumption of classically damped system the equation of motion can be decoupled by applying the modal analysis. To this aim let introduce the modal coordinate transformation

$$\hat{\mathbf{u}}(t) = \hat{\boldsymbol{\Phi}} \hat{\mathbf{q}}(t) = \sum_{k=1}^m \boldsymbol{\phi}_k q_k(t) \quad (22)$$

The hat means that the corresponding quantities are evaluated considering m modes ($m < n$ being a suitable integer). In Eq. 22 $\hat{\boldsymbol{\Phi}}$ is a matrix of order $m \times n$ collecting the m eigenvectors, normalized with respect to the nominal mass matrix $\mathbf{M}$, solution of the following eigenproblem

$$\mathbf{K}\hat{\boldsymbol{\Phi}} = \mathbf{M}\hat{\boldsymbol{\Phi}}\hat{\boldsymbol{\Omega}}^2; \quad \hat{\boldsymbol{\Phi}}^T \mathbf{M} \hat{\boldsymbol{\Phi}} = \mathbf{I}_m \quad (23)$$

The k -th modal shape $\boldsymbol{\phi}_k$ is evaluated along with the natural circular frequency ω_k by solving iteratively the eigenproblem. In Eq. 23 $\hat{\boldsymbol{\Omega}}$ is a diagonal matrix listing the natural circular frequencies ω_k .

By assuming the structural system classically damped, once the modal matrix $\hat{\boldsymbol{\Phi}}$ is evaluated, by applying the coordinate transformations (22) to Eq. 21, the following set of first order differential equations is obtained

$$\ddot{\hat{\mathbf{q}}}(t) + \hat{\boldsymbol{\Xi}} \dot{\hat{\mathbf{q}}}(t) + \hat{\boldsymbol{\Omega}}^2 \hat{\mathbf{q}}(t) = \hat{\mathbf{p}} \ddot{u}_g(t) \quad (24)$$

In which $\hat{\mathbf{p}}$ is the vector of participation coefficients p_k and $\hat{\mathbf{\Xi}}$ is the generalized damping matrix given respectively by

$$\hat{\mathbf{p}} = -\hat{\mathbf{\Phi}}^T \mathbf{M} \boldsymbol{\tau}; \quad \hat{\mathbf{\Xi}} = \hat{\mathbf{\Phi}}^T \mathbf{C} \hat{\mathbf{\Phi}}. \quad (25)$$

For classically damped structures the modal damping matrix $\hat{\mathbf{\Xi}}$ is a diagonal matrix listing the quantities $2\xi_k \omega_k$, being ξ_k the modal damping ratio that usually is assumed to be equal for all modes. It follows that the k -th differential equation can be written as

$$\ddot{q}_k(t) + 2\xi_k \omega_k \dot{q}_k(t) + \omega_k^2 q_k(t) = p_k \ddot{u}_g(t) \quad (26)$$

Remarkably, by the assumption of classically damped linear system the nodal response of the structural system can be determined by the superposition of the response of a number of single degree of freedom systems whose evaluation is nowadays straightforward.

In order to improve the accuracy of the response, evaluated by considering a reduced number of modes, a modal correction technique has to be introduced. Accordingly, the response vector of the linear systems subjected to seismic input can be written as the sum of two contributions

$$\mathbf{u}(t) = \hat{\mathbf{u}}(t) + \mathbf{u}_{\text{ps}}(t) \quad (27)$$

where $\hat{\mathbf{u}}(t)$ is the nodal response vector determined via the modal analysis considering the lower modes as the projection in the nodal space of the solution of Eq. 24. In Eq. 27 $\mathbf{u}_{\text{ps}}(t)$ is the pseudo-static corrective term that takes into account approximately the contribution of the higher neglected modes. In this section the mode-acceleration method (MAM) [4] is applied. The MAM is certainly the simplest and widely used method in both deterministic and stochastic cases; it improves the response vector by evaluating the corrective term in pseudo-static form, as

$$\mathbf{u}_{\text{ps}}(t) = -\left[\mathbf{K}^{-1} - \hat{\mathbf{\Phi}} \hat{\mathbf{\Omega}}^{-2} \hat{\mathbf{\Phi}}^T\right] \mathbf{M} \boldsymbol{\tau} \ddot{u}_g(t) = \mathbf{b} \ddot{u}_g(t) \quad (28)$$

In this equation $\mathbf{b} = -\left[\mathbf{K}^{-1} - \hat{\mathbf{\Phi}} \hat{\mathbf{\Omega}}^{-2} \hat{\mathbf{\Phi}}^T\right] \mathbf{M} \boldsymbol{\tau}$ is a vector which describes the spatial distribution of seismic loads taking into account the higher modes in pseudo-static form.

4 Stochastic Seismic Analysis

From a strict probabilistic point of view, the complete description of a stochastic process requires the evaluation of the probability density function. This function, in the case of linear systems subjected to ground motion, modeled as a Gaussian zero-mean process, is fully characterized by the knowledge of the correlation matrix, given by the following equation

$$\mathbb{E} \left\langle \hat{\mathbf{u}}(t_1) \hat{\mathbf{u}}^T(t_2) \right\rangle = \hat{\Phi} \mathbb{E} \left\langle \hat{\mathbf{q}}(t_1) \hat{\mathbf{q}}^T(t_2) \right\rangle \hat{\Phi}^T = \sum_{k=1}^m \sum_{\ell=1}^m \phi_k \phi_{\ell}^T \mathbb{E} \langle q_k(t_1) q_{\ell}(t_2) \rangle \quad (29)$$

where $\mathbb{E} \langle \bullet \rangle$ means stochastic average; then $\mathbb{E} \langle q_k(t_1) q_{\ell}(t_2) \rangle$ is the modal cross-correlation between the response of k -th and ℓ -th oscillators, governed by differential equations (26).

However, in many cases of engineering interest the reliability of the structural system is assessed. In these cases the probabilistic assessment of structural failure is derived as a function of barrier crossing rates, distribution of peaks and extreme values. These quantities can be evaluated as a function of the so-called spectral moments, introduced by Vanmarcke [33] for stationary stochastic processes. Application of spectral methods to non-stationary random processes is more complicated than for stationary processes, indeed for these processes the geometric approach fails [12, 25]. To solve this problem Di Paola [12] introduced a new complex-valued random process, the so-called “pre-envelope process”. For non-stationary processes, characterized by the one-sided evolutionary PSD function, these complex covariance was called non-geometric spectral moments by Michaelov et al. [23].

The non-geometric spectral moments of the i -th nodal response are real quantities, given by the following relationships

$$\begin{aligned} \hat{\lambda}_{0,u_i u_i}(t) &= \mathbb{E} \langle \hat{u}_i(t) \hat{u}_i(t) \rangle = \sum_{k=1}^m \sum_{\ell=1}^m p_k p_{\ell} \phi_{ik} \phi_{i\ell} \lambda_{0,k\ell}(t); \\ \hat{\lambda}_{1,u_i u_i}(t) &= \sum_{k=1}^m \sum_{\ell=1}^m p_k p_{\ell} \phi_{ik} \phi_{i\ell} \lambda_{1,k\ell}(t); \\ \hat{\lambda}_{2,u_i u_i}(t) &= \mathbb{E} \left\langle \hat{\dot{u}}_i(t) \hat{\dot{u}}_i(t) \right\rangle = \sum_{k=1}^m \sum_{\ell=1}^m p_k p_{\ell} \phi_{ik} \phi_{i\ell} \lambda_{2,k\ell}(t). \end{aligned} \quad (30)$$

Furthermore, in the stochastic analysis of the structures subjected to non-stationary input process, the “central frequency” and the “bandwidth parameter” are useful in describing the time-variant spectral properties of a generic real-valued non-stationary process. The central frequency provides the characteristic/predominant frequency of the process at each instant of time. The bandwidth parameter provides information on the spectral bandwidth of the process at each instant of time. Notice that a non-stationary process can behave as a narrowband and a broadband process at different instants of time. The “central frequency” $\hat{\omega}_{c,u_i u_i}(t)$ and the “bandwidth factor” $\hat{\delta}_{u_i u_i}(t)$ of the i -th nodal response can be evaluated as a function of non-geometric spectral moments as:

$$\hat{\omega}_{c,u_i u_i}(t) = \frac{\hat{\lambda}_{1,u_i u_i}(t)}{\hat{\lambda}_{0,u_i u_i}(t)}; \quad \hat{\delta}_{u_i u_i}(t) = \sqrt{1 - \frac{\hat{\lambda}_{1,u_i u_i}^2(t)}{\hat{\lambda}_{0,u_i u_i}(t) \hat{\lambda}_{2,u_i u_i}(t)}}. \quad (31)$$

As shown in Eq. 30 the nodal non-geometric spectral moments can be evaluated as a function of $\lambda_{i,k\ell}(t)$, $i = 0, 1, 2$, which are the so-called time-dependent non-geometric modal spectral moments “purged” by participation factors. These quantities are complex ones and for the spectrum compatible model defined in Eq. 20, according to non-geometric definition of spectral moments, can be evaluated in the frequency domain as follows

$$\begin{aligned}\lambda_{0,k\ell}(t) &= \sum_{j=0}^N \int_0^{\infty} H_{k,j}(\omega, t) H_{\ell,j}^*(\omega, t) G_j(\omega) d\omega; \\ \lambda_{1,k\ell}(t) &= -i \sum_{j=0}^N \int_0^{\infty} H_{k,j}(\omega, t) \dot{H}_{\ell,j}^*(\omega, t) G_j(\omega) d\omega; \\ \lambda_{2,k\ell}(t) &= \sum_{j=0}^N \int_0^{\infty} \dot{H}_{k,j}(\omega, t) \dot{H}_{\ell,j}^*(\omega, t) G_j(\omega) d\omega.\end{aligned}\quad (32)$$

It has to be emphasized that the zero-th and second order non-geometric spectral moments (32) coincide with the modal cross-covariance function of the response in terms of displacement and velocity, respectively [12, 14, 24–26]. Moreover, the presence of the imaginary unit in the second term of Eq. 32 inverts the roles of the real and imaginary parts of $\lambda_{1,k\ell}(t)$ with respect to the cross-covariance $\lambda_{0,k\ell}(t)$ and $\lambda_{2,k\ell}(t)$. Note that for $k = \ell$, $\lambda_{0,k\ell}(t)$ and $\lambda_{2,k\ell}(t)$ become real quantities, while $\lambda_{1,k\ell}(t)$ remain complex ones.

In Eq. 32 $H_{p,j}(\omega, t)$ and $\dot{H}_{p,j}(\omega, t)$, $p = k, \ell$, are the time-dependent complex function defined as

$$\begin{aligned}H_{p,j}(\omega, t) &= \exp(i\omega t) \int_0^t h_p(t - \tau) \exp(-i\omega\tau) a_j(\tau) d\tau. \\ \dot{H}_{p,j}(\omega, t) &= \exp(i\omega t) \int_0^t \dot{h}_p(t - \tau) \exp(-i\omega\tau) a_j(\tau) d\tau \neq \frac{\partial H_{p,j}(\omega, t)}{\partial t}.\end{aligned}\quad (33)$$

where

$$h_p(t) = \exp(-\xi_p \omega_p t) \sin\left(\omega_p \sqrt{1 - \xi_p^2} t\right) / \omega_p \sqrt{1 - \xi_p^2} \quad (34)$$

Remarkably, the integrals of Eq. 33 are convolution integrals of Duhamel’s type and they represent the response, in terms of state variables, of the p -th oscillator (26), purged by participation factor, subjected to deterministic complex function: $\exp(-i\omega t) a_j(t)$. Consequently traditional step-by-step procedure can be applied. Moreover, when the function $a_j(t)$ assumes mathematically simple expression, the integrals (33) can be evaluated in closed form.

5 Modal Correction Method for Stochastic Response

In the framework of stochastic analysis of multi-degree-of-freedom (MDOF) structures, modal analysis, along with modal truncation of higher modes, is extensively used to reduce the computational effort in determining the dynamic response of linear systems. The analysis is based on the projection of the equation of motion from the geometric (or nodal space) to the so-called generalized space, by using the eigenproperties of the undamped structural system. On the other hand, the computation effort involved in the analysis is further reduced neglecting the contribution of higher modes. However, the modal truncation criterion based on participation mass ratio, widely used for deterministic seismic analyses, is not directly applicable in the stochastic dynamics. It follows that to date it is not possible to predict *a priori* the number of modes necessary for an accurate evaluation of the stochastic response.

Moreover, the most efficient approach in the stochastic seismic analysis of large structural systems via modal analysis is nowadays based on the utilization of modal correction methods [1, 4, 10, 22]. These methods improve the response taking into account approximately the contribution of the neglected higher modes. In this framework the MAM, originally proposed by Maddox [21] for deterministic input, is the most simple and widely used method. In this section the extension of the MAM, to improve the stochastic response of structural systems subjected to fully non-stationary seismic input, is addressed.

In order to do this it is useful to remember preliminarily that, as recently showed by Cacciola et al. [4], the application of the MAM to the evaluation of the deterministic state variable (displacement and velocity) response leads to the following correction terms

$$\begin{aligned}\mathbf{u}_{\text{PS}}(t) &= -\left[\mathbf{K}^{-1} - \hat{\Phi} \hat{\Omega}^{-2} \hat{\Phi}^T\right] \mathbf{M} \tau \ddot{u}_g(t) = \mathbf{b} \ddot{u}_g(t); \\ \dot{\mathbf{u}}_{\text{PS}}(t) &= \mathbf{0}\end{aligned}\quad (35)$$

It follows that by applying the MAM, the displacements and velocity covariance matrices of the response can be written as

$$\begin{aligned}\mathbf{E}\left\langle \mathbf{u}(t) \mathbf{u}^T(t) \right\rangle &= \hat{\Phi} \mathbf{E}\left\langle \hat{\mathbf{q}}(t) \hat{\mathbf{q}}^T(t) \right\rangle \hat{\Phi}^T + \sigma_{\ddot{u}_g}^2 \mathbf{b} \mathbf{b}^T + \sum_{k=1}^m \phi_k \mathbf{b}^T \mathbf{E}\left\langle q_k(t) \ddot{u}_g(t) \right\rangle \\ &\quad + \sum_{\ell=1}^m \mathbf{b} \phi_\ell^T \mathbf{E}\left\langle \ddot{u}_g(t) q_\ell(t) \right\rangle \\ \mathbf{E}\left\langle \dot{\mathbf{u}}(t) \dot{\mathbf{u}}^T(t) \right\rangle &= \hat{\Phi} \mathbf{E}\left\langle \dot{\hat{\mathbf{q}}}(t) \dot{\hat{\mathbf{q}}}^T(t) \right\rangle \hat{\Phi}^T\end{aligned}\quad (36)$$

where the stochastic averages which appear in the last three terms on the right-hand side of the first of Eq. 36 are given by

$$\begin{aligned}\sigma_{\ddot{u}_g}^2 &= \sum_{j=0}^N a_j^2(t) \int_0^\infty G_j(\omega) d\omega; \\ E\langle q_k(t) \ddot{u}_g(t) \rangle &= p_k \sum_{j=0}^N a_j(t) \int_0^\infty H_{k,j}(\omega, t) G_j(\omega) \exp(i\omega t) d\omega; \\ E\langle \ddot{u}_g(t) q_\ell(t) \rangle &= p_\ell \sum_{j=0}^N a_j(t) \int_0^\infty H_{\ell,j}^*(\omega, t) G_j(\omega) \exp(-i\omega t) d\omega. \quad (37)\end{aligned}$$

Note that, since the one-sided power spectral density has been introduced [5], the functions $E\langle q_k(t) \ddot{u}_g(t) \rangle$ and $E\langle \ddot{u}_g(t) q_\ell(t) \rangle$ are complex conjugate quantities, so that the first of Eq. 36 leads to a real covariance matrix. Remarkably, through Eq. 36 the accuracy in determining the covariance matrix can be easily improved with a slight increment of computational effort.

By applying the MAM to non-geometric spectral moments, after very simple algebra, the following equations are obtained:

$$\begin{aligned}\lambda_{0,u_i u_i}(t) &= \sum_{k=1}^m \sum_{\ell=1}^m p_k p_\ell \phi_{ik} \phi_{i\ell} \lambda_{0,k\ell}(t) + b_i^2 \sigma_{\ddot{u}_g}^2 + 2b_i \sum_{k=1}^m \phi_{ik} \operatorname{Re} \{E\langle q_k(t) \ddot{u}_g(t) \rangle\}; \\ \lambda_{1,u_i u_i}(t) &= \sum_{k=1}^m \sum_{\ell=1}^m p_k p_\ell \phi_{ik} \phi_{i\ell} \lambda_{1,k\ell}(t) + b_i \sum_{\ell=1}^m \phi_{i\ell} \operatorname{Im} \{E\langle \ddot{u}_g(t) \dot{q}_\ell(t) \rangle\}; \\ \lambda_{2,u_i u_i}(t) &\equiv \hat{\lambda}_{2,u_i u_i}(t) = E\langle \hat{u}_i(t) \hat{u}_i(t) \rangle = \sum_{k=1}^m \sum_{\ell=1}^m p_k p_\ell \phi_{ik} \phi_{i\ell} \lambda_{2,k\ell}(t). \quad (38)\end{aligned}$$

where

$$E\langle \ddot{u}_g(t) \dot{q}_\ell(t) \rangle = p_\ell \sum_{j=0}^N a_j(t) \int_0^\infty \dot{H}_{\ell,j}^*(\omega, t) G_j(\omega) \exp(-i\omega t) d\omega. \quad (39)$$

is a complex function which defines the input–output relationship [5]. It is noted that the MAM leads to a correction to zero-th order and first order non-geometric response spectral moments, while it is not able to correct the second order non-geometric response spectral moments. This is owed to the fact the MAM proposed for the deterministic case is able to correct only the response displacements and not the velocity as shown also in Eq. 38.

6 Numerical Results

The procedure described in previous sections is herein applied to the structure showed in Fig. 1. The structure possesses 722 frame elements with 819 degrees of freedom. Mass is assumed lumped at each node. Furthermore, the structure is assumed classically damped with a modal damping ratio $\xi_0 = 0.05$ assumed constant for each mode. Pertinent first three modes are reported in Fig. 2a–c. The structure undergoes to forced vibration due a ground motion acceleration acting in x-direction modeled as a Gaussian fully non-stationary spectrum compatible process. The procedure proposed by Cacciola [2] and described in Sect. 2.2 has been used for this purpose. To this aim El Centro 1940 accelerogram has been selected for illustrative purpose for modeling the spectrum compatible process as fully non stationary and consistent with a given target response spectrum as well.

Specifically, EC8 soil type B spectrum has been selected as target response spectrum. In Fig. 3 it is represented the pertinent evolutionary spectrum compatible power spectral density function determined through Eq. 12. The parameters N , α_j , β_j , γ_j , ϑ_j , η_j , v_j ($j = 1, \dots, N$), for the function $G_{u_g}^R(\omega, t)$ defined in Eq. 13, are given in [7]; while the modulating function $a_0(t)$, which characterizes the function $G_{u_g}^S(\omega, t)$ introduced in Eq. 16, is given by the following equation [20]

$$a_0(t) = \begin{cases} \left(\frac{t}{t_1}\right)^2 & t < t_1 \\ 1 & t_1 \leq t \leq t_2 \\ \exp[-\beta(t - t_2)] & t > t_2 \end{cases} \quad (40)$$

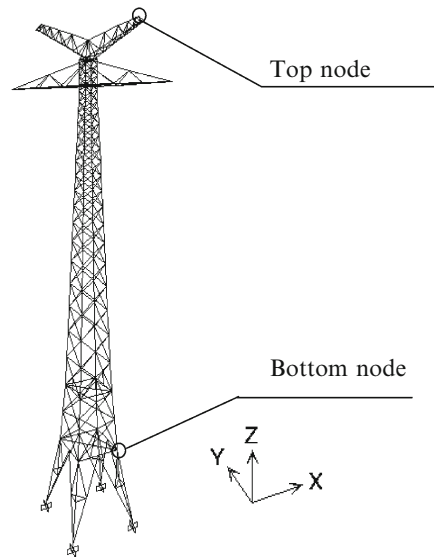


Fig. 1 Structural model

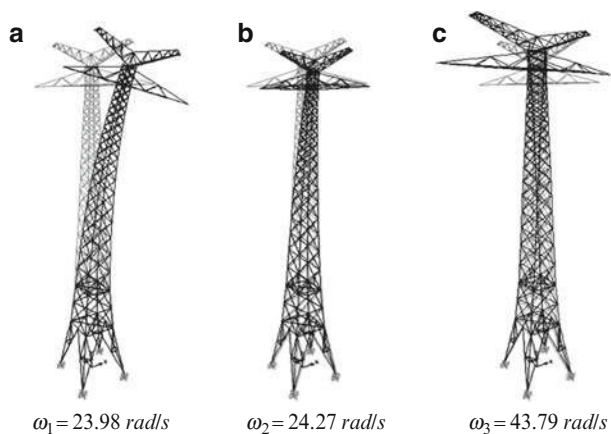


Fig. 2 First three modes: (a) first mode, (b) second mode, (c) third mode

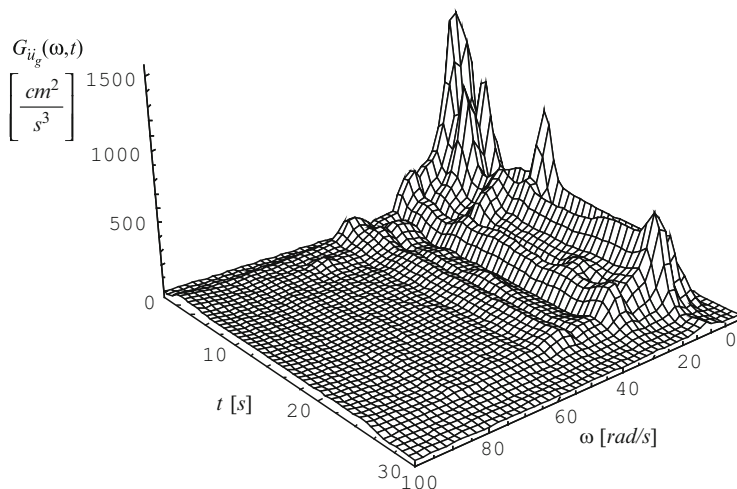


Fig. 3 Evolutionary spectrum compatible power spectral density function

with $t_1 = 1.646$ s; $t_2 = 25.51$ s and $T_s = 23.86$ s determined by means of the Husid [18] function of the El Centro 1940 earthquake, so respecting the limits of $T_s > 10$ s provided by Eurocode 8. Furthermore, β has been set equal to $3/(t_f - t_2)$ so reducing the amplitude of the simulated signal of about 95% for $t = t_f$. The time variability of both amplitude and frequency are manifested by the study of the mean instantaneous energy $\langle E(t) \rangle$ and central frequency of input process $\langle \omega(t) \rangle$, given respectively by the following equations (see e.g. [31])

$$\langle E(t) \rangle = \int_0^{\infty} G_{\ddot{u}_g}(\omega, t) d\omega \quad (41)$$

and

$$\langle \omega(t) \rangle = \frac{\int_0^{\infty} \omega G_{\ddot{u}_g}(\omega, t) d\omega}{\int_0^{\infty} G_{\ddot{u}_g}(\omega, t) d\omega} \quad (42)$$

Figures 4 and 5 show the time variability of the mean instantaneous energy and central frequency manifesting the fully non-stationary behavior of the spectrum compatible model.

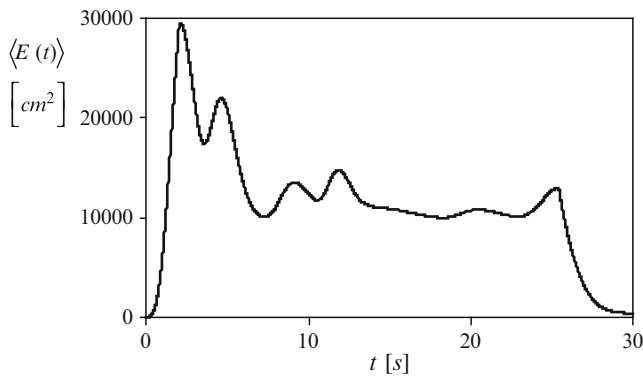


Fig. 4 Mean instantaneous energy of the spectrum compatible model

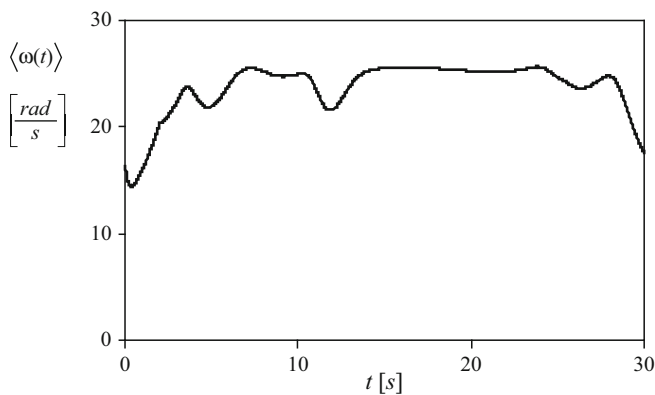


Fig. 5 Mean instantaneous central frequency of the spectrum compatible model

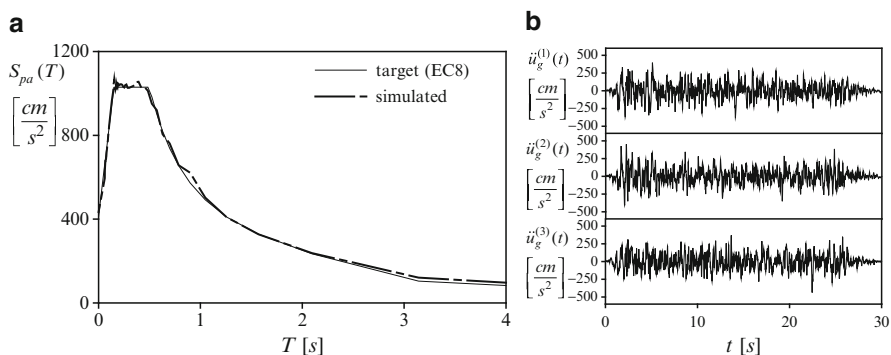


Fig. 6 Comparison between simulated and target response spectra (a) and spectrum compatible time-histories (b)

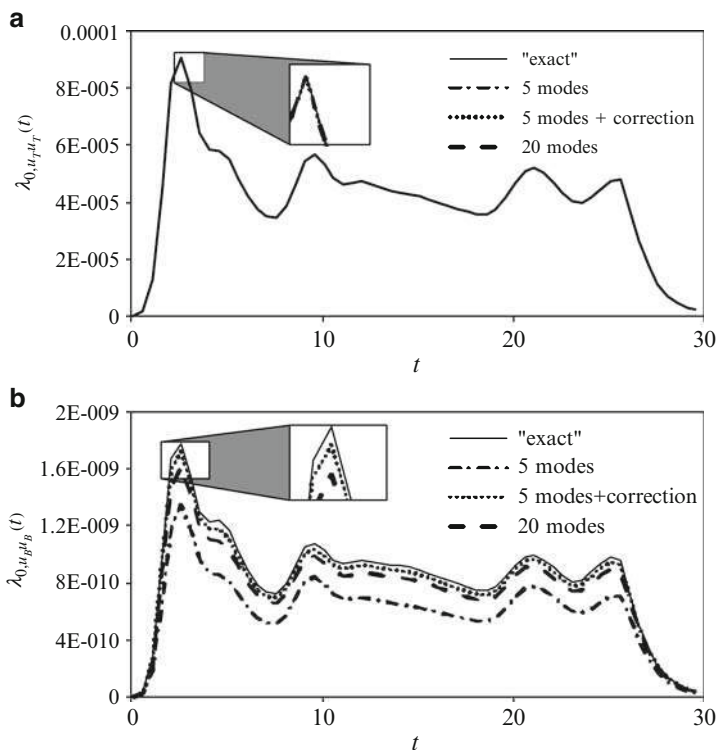


Fig. 7 Zero-th order non-geometric response spectral moments (a) top (b) bottom

Spectrum compatible criteria are verified in Fig. 6 by the positive matching of the simulated and target response spectra.

Once the seismic action has been defined, the nonstationary response has been determined by the proposed procedure described in the previous sections. It has

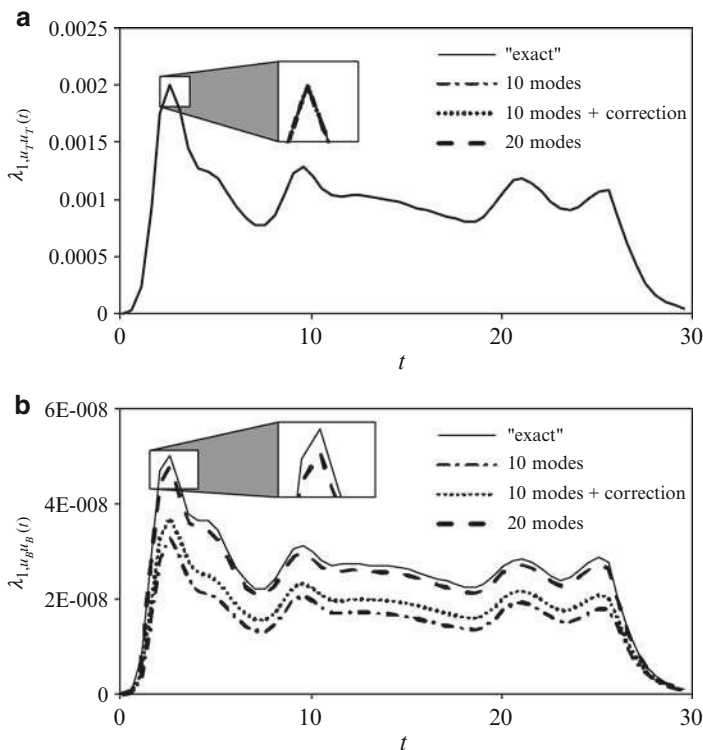


Fig. 8 First order non-geometric response spectral moments (a) top (b) bottom

to be emphasized that the adopted fully nonstationary spectrum compatible model allows the evaluation of function $H_k(\omega, t)$ defined in Eq. 33 in closed form, making the proposed procedure very competitive from a computational point of view.

Figures 7–9 show the non-stationary non-geometric spectral moments of the response for two selected nodes of the structure, as indicated in Fig. 1. By the analysis of the figures appears that the modal correction method give a sensible contribution to the accuracy of the bottom node, whose response is affected by higher order modes. On the other hand the response of top node is accurately determined without the need of a correction being governed by first two lower modes only. As expected according to the philosophy of MAM the correction method is efficient for correcting the displacement covariances, which is for the zero-th non-geometric spectral moments (Fig. 7), while its efficiency deteriorates when velocities are involved (see the first order and second order non-geometric spectral moments of Figs. 8 and 9).

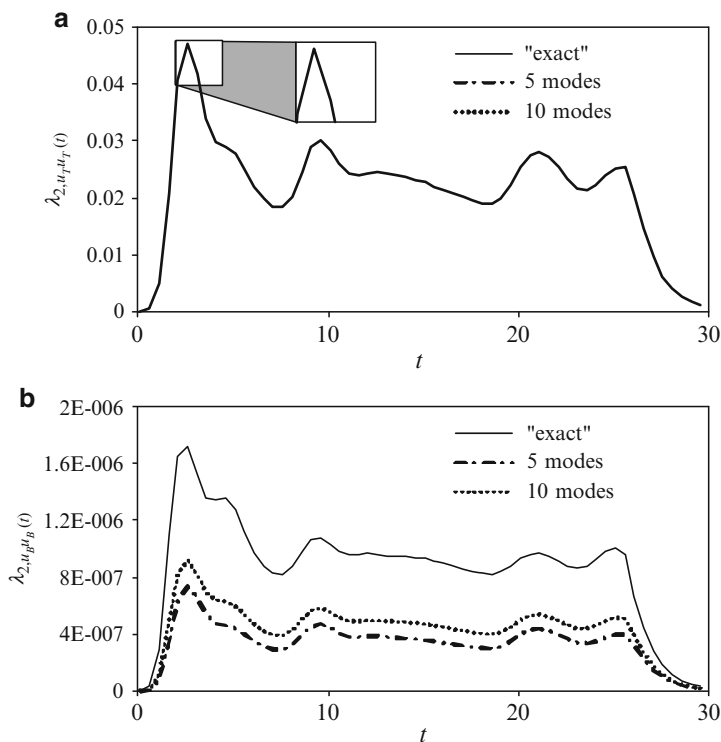


Fig. 9 Second order non-geometric response spectral moments (a) top (b) bottom

7 Concluding Remarks

A new computationally competitive method for the seismic analysis of linear system vibrating under spectrum compatible fully non-stationary Gaussian excitations has been proposed in this chapter. The method requires the following steps:

- The definition of the spectrum compatible process as the sum of two independent counterparts, the first associated to a recorded accelerogram, which takes into account of the time variability of both intensity and frequency content; while the second one is a corrective random term adjusting the spectrum compatibility.
- The use of modal analysis to decouple the equations of motion of classically damped systems.
- The evaluation of the modal non-stationary non-geometric spectral moments by handy formulas, computationally very efficient in the case of non-separable power spectral densities.

In order to improve the accuracy of the nodal non-geometric spectral moments of the response an extension of the mode-acceleration method (MAM) for the evaluation of the non-stationary spectral moments has been presented.

Remarkably, the use of modal correction technique in the framework of stochastic seismic analysis, appears to date the most suitable strategy for accurately determine the random response despite a slight increment of computational effort compared to the adding of higher modal responses in evaluating the statistics of the response. Furthermore, it is noted that modal truncation criteria defined for the deterministic case cannot be used directly in the stochastic analysis. As a consequence modal correction techniques offer a valuable strategy for coping with the uncertainty related to the number of modes to be selected.

Numerical analyses conducted on a large MDOF system vibrating under spectrum compatible excitation models showed the accuracy and the efficiency of the proposed procedure in order to define the spectrum compatible stochastic seismic input and in the evaluation of the modal non-geometric non-stationary spectral moments.

It has been shown also that the classical MAM is very convenient for determining the zero-th order response spectral moments with a slight additional computational effort. On the other hand this technique cannot improve the accuracy of second order response spectral moments because the latter involve the response velocities only. Further studies have been still devoted to correction methods in order to meaningfully improve the accuracy of spectral moments involving the velocity of the response.

References

1. Benfratello, S., Muscolino, G.: Mode-superposition correction method for deterministic and stochastic analysis of structural systems. *Comput. Struct.* **79**, 2471–2480 (2001)
2. Cacciola, P.: A stochastic approach for generating spectrum compatible fully nonstationary earthquakes. *Computers & Structures* **88**(15–16), 889–901 (2010)
3. Cacciola, P., Colajanni, P., Muscolino, G.: Combination of modal responses consistent with seismic input representation. *J. Struct. Eng.* **130**(1), 47–55 (2004)
4. Cacciola, P., Maugeri, N., Muscolino, G.: A modal correction method for non-stationary random vibrations of linear systems. *Probab. Eng. Mech.* **22**, 170–180 (2007)
5. Cacciola, P., Muscolino, G.: Stochastic analysis of large structural systems under fully non-stationary input. 9th World Conference on Computational Mechanics, WCCM 2010, Sydney, Australia, 19–23 July 2010
6. Cohen, L.: Time–frequency distributions – a review. *Proc. IEEE* **77**(7), 941–981 (1989)
7. Conte, J.P., Peng, B.F.: Fully nonstationary analytical earthquake ground-motion model. *J. Eng. Mech.* **123**(1), 15–24 (1997)
8. Davenport, A.G.: Note on the distribution of the largest value of a random function with application to gust loading. *Proc Inst. Civil Eng.* **28**, 187–196 (1964)
9. Der Kiureghian, A.: Structural response to stationary oscillation. *J. Eng. Mech. (ASCE)* **106**, 1195–1213 (1980)
10. Der Kiureghian, A., Nakamura, Y.: CQC modal combination rule for high-frequency modes. *Earthquake Eng. Struct. Dyn.* **22**, 943–956 (1993)
11. Der Kiureghian, A., Neuenhofer, A.: A response spectrum method for multiple-support seismic excitation. Report No. UCB/EERC-91/08. Earthquake Engineering Research Center. University of California, Berkeley, CA (1991)

12. Di Paola, M.: Transient spectral moments of linear systems. *SM Arch.* **10**, 225–243 (1985)
13. Di Paola, M., La Mendola, L.: Dynamics of structures under seismic input motion (Eurocode 8). *Eur. Earthquake Eng.* **2**, 36–44 (1992)
14. Di Paola, M., Petrucci, G.: Spectral moments and pre-envelope covariances of nonseparable processes. *J. Appl. Mech. (ASME)* **57**, 218–224 (1990)
15. European Committee for Standardization, Eurocode 8: Design of structures for earthquake resistance – Part 1: General rules, seismic actions and rules for buildings, Brussels, Belgium (2003)
16. Falsone, G., Neri, F.: Stochastic modelling of earthquake excitation following the EC8: power spectrum and filtering equations. *Eur. Earthquake Eng.* **1**, 3–12 (2000)
17. Gupta, A.K.: *Response Spectrum Method in Seismic Analysis and Design of Structures*. Blackwell, Cambridge, MA (1990)
18. Husid, R.L.: Analisis de terremotos – Analisis general. *Revista del IDEM* **8**, 21–42 (1969). Santiago del Chile
19. International Conference of Building Officials: Uniform Building Code, vol. 2: Structural Engineering Design Provisions (1997)
20. Jennings, P.C., Housner, G.W., Tsai, C.: Simulated earthquake motions for design purpose. *Proc. 4th World Conf. Earth. Eng. Santiago A-1*, 145–160 (1969)
21. Maddox, N.R.: On the number of modes necessary for accurate response and resulting forces in dynamic analyses. *J. Appl. Mech. (ASME)* **42**, 516–517 (1975)
22. Maldonado, G.O., Singh, M.P., Suarez, L.E.: Random response of structure by a force derivative approach. *J. Sound Vib.* **155**, 13–29 (1992)
23. Michaelov, G., Sarkani, S., Lutes, L.D.: Spectral characteristics of nonstationary random processes – a critical review. *Struct. Saf.* **21**, 223–244 (1999)
24. Michaelov, G., Sarkani, S., Lutes, L.D.: Spectral characteristics of nonstationary random processes – response of a simple oscillator. *Struct. Saf.* **21**, 245–244 (1999)
25. Muscolino, G.: Nonstationary envelope in random vibration theory. *J. Eng. Mech. (ASCE)* **114**, 1396–1413 (1988)
26. Muscolino, G.: Nonstationary pre-envelope covariances of nonclassically damped systems. *J. Sound Vib.* **149**, 107–123 (1991)
27. Preumont, A.: A method for the generation of artificial earthquake accelerograms. *Nucl. Eng. Des.* **59**, 357–368 (1980)
28. Preumont, A.: The generation of nonseparable artificial earthquake accelerograms for the design of nuclear power plants. *Nucl. Eng. Des.* **88**, 59–67 (1985)
29. Priestley, M.B.: *Spectral Analysis and Time Series*. Academic, New York, London (1981)
30. Shinozuka, M.: Stochastic fields and their digital simulation, stochastic methods. In: Schueller, G.I., M. Shinozuka (eds.) *Structural Dynamics*, pp. 93–133. Martinus Nijhoff, Dordrecht (1987)
31. Spanos, P.D., Politis, N.P., Thomaidis, P.M.: Time-Frequency localization in stochastic seismic models via time-frequency atoms. 9th ASCE Specialty Conference on Probabilistic Mechanics and Structural Reliability, Albuquerque, New Mexico, 26–28 July 2004
32. Spanos, P.D., Vargas Loli, L.M.: A statistical approach to generation of design spectrum compatible earthquake time histories. *Soil Dyn. Earthquake Eng.* **4**(1), 2–8 (1985)
33. Vanmarcke, E.H.: Properties of spectral moments with applications to random vibration. *J. Eng. Mech. (ASCE)* **98**, 425–446 (1972)
34. Vanmarcke, E.H., Gasparini D.A.: Simulated earthquake ground motions. In: *Proceedings of the 4th International Conference on Smirt, K1/9*, San Francisco (1977)
35. Wang, J., Fan, L., Qian, S., Zhou, J.: Simulations of non-stationary frequency content and its importance to seismic assessment of structures. *Earthquake Eng. Struct. Dyn.* **31**, 993–1005 (2002)
36. Wang, J., Zhou, J.: Aseismic design based on artificial simulations. *IEEE Signal Process Mag.* **16**, 94–99 (1999)
37. Yeh, C.H., Wen, Y.K.: Modeling of non-stationary ground motion and analysis of inelastic structural response. *Struct. Saf.* **8**(1–4), 281–298 (1990)

Soil Spatial Variability and Structural Reliability of Buried Networks Subjected to Earthquakes

Sidi Mohammed Elachachi, Denis Breyse, and Hichem Benzeguir

Abstract Dysfunctions and failures of buried pipe networks like sewer networks are studied from the point of view of the heterogeneity of geotechnical conditions in the longitudinal direction and of the applied action (seismic action). Combined soil defects (differential settlements along the pipe, landslides, voids surrounding the pipe, etc.) and peak ground acceleration (PGA) induce stresses (which leads to an ultimate limit state ULS) and displacements (which constitute a violation of a serviceability limit state SLS). It is remarkable to note that the influence of the variability of the soil is not reflected in current European standards. A model has been developed which includes a description of the soil spatial variability, within the frame of geostatistics, where the correlation length of soil properties is the main parameter and a mechanical description of the soil–structure interaction of a set of buried pipes with flexible connections resting on the soil by a two parameter model (Pasternak model). Reliability analysis is performed on the sewer by using a Response Surface Model (RSM), with the reliability index calculated for two limit states: Serviceability limit state, corresponding to a too large “counterslope” in a given pipe, which can prevent the normal flow of fluids, and Ultimate limit state, corresponding to a too large bending moment, thus bending stress, which can cause cracks in the pipes. The response in time domain of a buried pipe subjected to natural ground motion records and by taking into account a longitudinal variability of the properties of the soil is modeled. Several conclusions are drawn: Soil heterogeneity induces effects (differential settlements, bending moments, stresses and possible cracking) that cannot be predicted if homogeneity is assumed and the magnitude of the induced stresses depends mainly on four factors: the soil-structure length ratio, which combines the soil fluctuation scale and a structural characteristic

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length (buried pipe length), a magnitude of the soil variability (i.e. its coefficient of variation), a soil-structure stiffness ratio, and a structure-connection stiffness ratio (relative flexibility).

1 Introduction

Pipes that carry different substances need to be designed so as to reduce the damage caused by ground displacement induced by earthquake. Heavy damage to pipelines has occurred in many strong earthquakes. Dysfunctions and failures of buried pipe networks like sewer networks are mainly due to the heterogeneity of geotechnical conditions in the longitudinal direction and of the applied action (seismic action). Combined soil defects (differential settlements along the pipe, landslides, voids surrounding the pipe, etc.) and peak ground acceleration (PGA) induce stresses (which leads to an ultimate limit state ULS) and displacements (which constitute a violation of a serviceability limit state SLS). It is remarkable to note that the influence of the variability of the soil is not reflected in current European standards [1].

Manolis and Beskos [2] have presented a comprehensive review on the seismic behavior of buried pipelines and underground structures and many references can be found there. In this paper the response of buried pipe is investigated considering the seismic excitation by selecting real earthquake data.

2 Pipe Modeling

2.1 Pasternak Model

Among the models which describe the behavior of a beam resting on a soil and their interaction, Pasternak model [3] constitutes an interesting model. In the soil-pipe interaction, the soil opposes, on the components of a sewer network (pipes), a distributed force $R(x)$ (in N/m) given by Eq. 1:

$$R(x) = p(x) \cdot D_{ext} \quad (1)$$

where $p(x)$ is the stress (under the pipe in Pa), and D_{ext} the external diameter of the pipe (in m). The stress is expressed according to the Pasternak model as:

$$p(x) = k_w \cdot w(x) - k_s \frac{d^2 w}{dx^2} \quad (2)$$

where k_w is the coefficient of subgrade reaction (or Winkler coefficient in N/m³ or Pa/m), k_s is the shear coefficient in N/m and $w(x)$ the vertical pipe displacement (and thus the settlement of the soil).

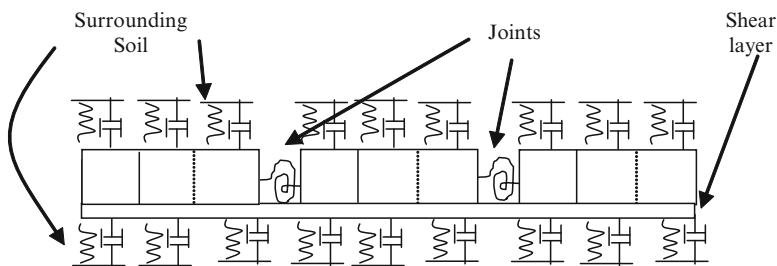


Fig. 1 Buried pipe rheological model

Pasternak's idealization considers the soil as being a system of identical but mutually independent, closely spaced, discrete, linearly elastic springs related by an incompressible “shear layer” which is defined as a layer of linear-elastic material of unit thickness that resists vertical shear forces only (Fig. 1). Thus it is a refinement of the well known Winkler model which suffers from not describing shear influence. The use of a two parameter model to characterize the soil's response under loading can appear as a too simplified concept. However, such a simplification seems coherent if one takes into account the variability and uncertainties related to the characterization of a soil. The spatial correlations which will be introduced further will ensure, in fact, a coherence of displacements, like it exists in a continuous medium.

One should note that the two coefficients k_w and k_s are not soil specific parameters. They are also affected by the rigidity of the pipe. In fact these stiffness parameters depend on several factors, such as the length and/or the width (or diameter) of the pipe, the laying depth, the type of material used and the type of the pipe bed. The value of these coefficients can only be approached by semi-empirical methods. For a same set of values, a parametric study was conducted [4] and illustrated the fact that for example k_w range varies from 1 to 3 according to authors. To the knowledge of the authors, the literature has not given ways on how to identify k_s in practice from physical and geometrical data [5].

In order to integer the damping of the system, the Pasternak model is adapted and extended to a Kelvin-Voigt model by adding dashpots (Fig. 1).

2.2 Stiffness Matrix

The stiffness matrix of the soil–pipe system has been derived in [4, 6] by using the energy method. Thus, the stiffness element matrix is given by the following expressions:

$$[K^e] = [K_b^e] + [K_w^e] + [K_s^e] \quad (3)$$

where:

$$[K_b^e] = \frac{E_p I}{l_e^3} \begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix} [K_s^e] = \frac{k_s D_{\text{ext}}}{30.l_e} \begin{bmatrix} 36 & 3l_e & -36 & 3l_e \\ 3l_e & 4l_e^2 & -3l_e & -l_e^2 \\ -36 & -3l_e & 36 & -3l_e \\ 3l_e & -l_e^2 & -3l_e & 4l_e^2 \end{bmatrix} \quad (4a)$$

and:

$$[K_w^e] = \frac{k_w D_{\text{ext}} l_e}{420} \begin{bmatrix} 156 & 22l_e & 54 & -13l_e \\ 22l_e & 4l_e^2 & 13l_e & -3l_e^2 \\ 54 & 13l_e & 156 & -22l_e \\ -13l_e & -3l_e^2 & -22l_e & 4l_e^2 \end{bmatrix} \quad (4b)$$

where E_p , I , D_{ext} , L and l_e are respectively the pipe Young modulus, the pipe moment of inertia, the exterior diameter, the pipe length and the finite element length.

The relative soil-pipe stiffness ratio r_p which is defined by the expression (5) can be considered as a governing parameter:

$$r_p = \frac{1}{L} \sqrt[5]{\frac{E_p I}{k_w}} \quad (5)$$

In order to be able to take into account the shear effect in a non-dimensional way, we introduce the following parameter r_{ks} which relates the Winkler coefficient k_w to the shear coefficient k_s .

$$r_{ks} = \frac{k_s}{k_w D_{\text{ext}}^2} \quad (6)$$

2.3 Joint-Connection Stiffness

Sealing between pipes is ensured by joints made of cement mortar or more frequently of elastomer. The rigidity of these joints is as variable as technologies and geometries employed: it can be very weak (flexible joints) or very high (welded joints). It is difficult to identify realistic numerical values of the joint stiffness even if some laboratory experiments have focused on this question [7]. In this work we assume a continuity of vertical displacements at joint connection and contrary to a model of continuous beam; we developed a model which enables to introduce discontinuities of rotation between the ends of the pipes. The joints between two adjacent pipes are assimilated to rotation springs with stiffness R_j relating the proportionality between the bending moment M applied to the joint and the variation of the angle of rotation $\Delta\varphi$:

$$M = R_j \cdot \Delta\varphi \quad (7)$$

To take into account this joint stiffness, we introduce the pipe-joint stiffness ratio r_{joint} which is defined by:

$$r_{\text{joint}} = \frac{R_j L}{E_p I} \quad (8)$$

To summarize, the soil-pipe interaction system, one can say that it is governed by a geometric parameter L (pipe's length) and three relative (without dimension) stiffness ratio parameters:

r_{ks} (compression to shear)

r_p (soil to pipe)

and r_{joint} (connection to pipe)

3 Modeling the Spatial Variability of Soils and Its Effects: The Correlation Length

In many common geotechnical problems, the variability of soils is only one among many sources of uncertainty (others are for example reduced sampling measurement or model errors). It can be accounted for by taking conservative values of the soil parameters, even if geotechnicians need a long practice to be able to justify the choice of these values [8]. In fact, the variability of soils cannot be reduced to case-by-case variability – as a result of its natural or man-made fabric (deposit processes, compaction processes...), the soil properties can be considered as spatially structured. Thus, tools like autocorrelation functions or semi-variograms appear to be appropriate tools for modeling. One must then identify the standard deviation or coefficient of variation of the studied property as well as the correlation length l_c (i.e. the distance above which the local properties at two points can be assumed to be independent).

The first consequence of the spatial correlation is that the representative value of any soil property depends on the volume concerned by the problem to be solved. This question has been analyzed in detail during the drafting of Eurocode-7 but code writers have limited themselves to general considerations, without prescribing any formal method: the representative value is only said to be a characteristic value defined as “a cautious estimate of the parameter governing the studied limit state” [9].

Nevertheless, accounting for spatial correlation has direct consequences on the safety of designs. A simple illustration of the effects of spatial variability is that of the rotation (tilting) of a foundation of length L resting on a heterogeneous elastic soil and supporting a uniform loading. It was shown [10] that the magnitude of the rotation depends on the correlation length in the horizontal direction: it tends towards zero when l_c is very small (which corresponds to very quickly varying properties, thus homogeneous at L scale) or very large (which corresponds to very slowly varying properties, thus also homogeneous at L scale) and it is maximum for an intermediate range. This illustrates a consequence of the spatially correlated variation of soil properties: tilting occurs only when the soil is not homogeneous

and its magnitude depends both on the scatter in the soil properties (linearly) and on the correlation length, with a ‘worst case’ for a particular range of l_c values.

Since the role of the longitudinal variability of the filling appears essential, we chose to model it by using the theory of the local average of a random field developed by VanMarcke [11].

The random field of the coefficients k_w or k_s is defined by three properties: its average value $\overline{k_w}$ (resp. $\overline{k_s}$), its variance $\overline{k_w}^2$ (resp. $\overline{k_s}^2$) and its scale (or length) of correlation l_c (resp. l_{cs}). These scales are related to a function (Eq. 9) of correlation $\rho(\tau)$, where τ points out the distance between two points, and which describes the spatial structure of correlation of the properties: $\rho(\tau)$ differs whether the properties vary more or less quickly while deviating from a given point. This correlation length (length from which the correlation between soil properties tend to disappear) depends on the characteristic (modulus, porosity, water content...) and on the direction (horizontal or vertical).

$$\rho(\tau) = \exp\left(-2\frac{|\tau|}{l_c}\right) \text{ for } \tau \leq l_c \quad (9)$$

The soil is subdivided in several zones. The random field value in each zone is thus a random variable whose value is estimated by the average of the space field over the zone. The local average and the variance in zone i of length D_i fulfil (expression 10):

$$E[k_w(D_i)] = \overline{k_w} \quad (10a)$$

$$\text{Var}[k_w(D_i)] = \overline{k_w}^2 \gamma(D_i) \quad (10b)$$

The expression (10b) shows that the local variance $\text{var}[k_w(D_i)]$ depends on the length D_i of zone i while following a variance reduction function $\gamma(D_i)$.

$$\gamma(D_i) = 2 \left(\frac{l_c}{D_i}\right)^2 \left(\frac{D_i}{l_c} - 1 + \exp\left(-\frac{D_i}{l_c}\right)\right) \quad (11)$$

$\gamma(D_i)$ is a measure of the variance reduction due to the averaging of the random process according to the length of the studied zone. In this work, as well k_w as k_s follow a Lognormal distribution.

In order to show the effect of both the mean values of the Winkler and shear coefficients and their correlation length, six cases were considered (Table 1) and applied to an example of a sewer section (set defined between two manholes) and which will be presented in detail further. The section is made of 20 pipes of unit length $L = 3$ m. For each computation, outputs which are linked to limit states are processed all along the section and their worst value is kept. Finally the cumulative density function (cdf) of this parameter (stress, displacement, bending moment) is drawn and used for comparisons.

Table 1 Values of k_w , k_s , l_c and l_{cs}

	k_w (kN/m ³)	l_c (m)	k_s (kN/m)	l_{cs} (m)
Case A	10	3	1	30
Case B	10	30	1	3
Case C	10	3	1	3
Case D	10	3	10	3
Case E	10	30	10	3
Case F	10	3	10	30

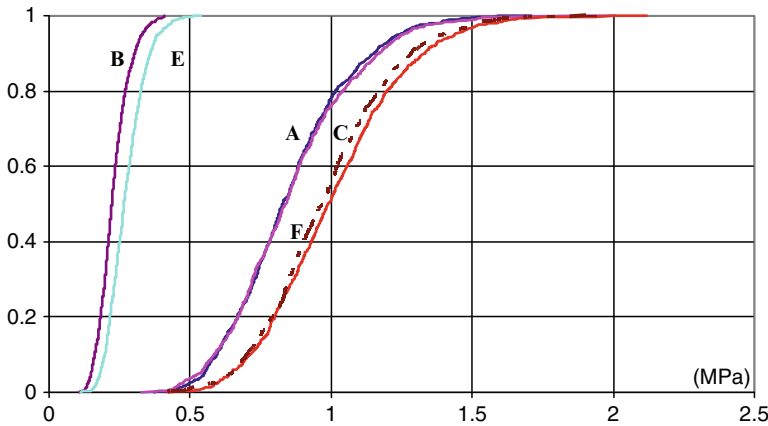
**Fig. 2** Cdf of bending stress for the six different cases presented in Table 1

Figure 2 shows the cdf of the maximum bending stress in a concrete sewer's section. One can make two observations:

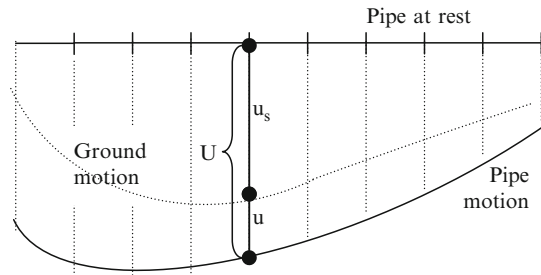
- To not take into account the shear effect is not conservative (case C compared to case F) being given that the mean stress varies from more than 20%,
- The correlation length of Winkler coefficient seems to have more importance than the correlation length of the shear coefficient (case B compared to A and C cases or case E compared to D and F cases).

4 Equations of Motion

The soil resistance to the pipe motion is generated by the relative motion u between the pipe and the soil. The pipe resistance comes from the absolute displacement U (Fig. 3). The governing equation of the system is as follows [12]:

$$\begin{aligned}
 & [M_p + M_{\text{Soil}}] \{\ddot{U}\} + [C_p + C_{\text{Soil}}] \{\dot{U}\} + [K_p + K_w + K_s] \{U\} \\
 & = [M_{\text{Soil}}] \{\ddot{u}_s\} + [C_{\text{Soil}}] \{\dot{u}_s\} + [K_w + K_s] \{u\}
 \end{aligned} \tag{12}$$

Fig. 3 Pipe and ground motion



In which M_p and M_{soil} are respectively the mass matrix of the pipe and of the soil, K_p , K_w and K_s the stiffness matrix of the pipe and the soil (defined in Sect. 2.2), C_p and C_{soil} respectively the damping matrices of the pipe and of the soil. We note that the damping matrices are constructed from the Rayleigh damping. U , $\dot{U}$, $\ddot{U}$, are respectively vectors of absolute displacement, velocity and acceleration, u , $\dot{u}$ and $\ddot{u}$ are vectors of relative displacement, velocity and acceleration, u_s , $\dot{u}_s$ and $\ddot{u}_s$ are vectors of ground displacement, velocity and acceleration.

Denoting the total stiffness $[K] = [K_p] + [K_w] + [K_s]$, the total Mass $[M] = [M_p] + [M_{soil}]$ and the total damping $[C] = [C_p] + [C_{soil}]$, the absolute displacement $\{U\} = \{u_s\} + \{u\}$, the equations of motion in terms of the relative displacement can be rewritten as:

$$[M] \{\ddot{u}\} + [C] \{\dot{u}\} + [K] \{u\} = -[M_p] \{\ddot{u}_s\} - [C_p] \{\dot{u}_s\} - [K_p] \{u_s\} \quad (13)$$

These equations of motion are solved for each time history of the ensemble of real earthquake records. The boundary conditions can be considered as well fixed ends or free ends if we want to eliminate their effects.

5 Response Surface Method

In general, Monte Carlo Simulation (MCS) is suitable for reliability analyses of engineering structures, but a large number of samples is required. Several much more efficient methods have been developed to reduce this aspect, which are improved Monte Carlo simulations like importance sampling or adaptive sampling. Another powerful and widely used approach to such an approximation is the Response Surface Method (RSM).

Response Surface Method refers not simply to the use of a response surface as a multivariate function but also to the process for determining the polynomial coefficients themselves. The basic idea of the RSM is to approximate the implicit limit state function $G(x)$ by an equivalent polynomial function $G_{equ}(x)$. Thanks to this approximation, the reliability assessment becomes faster and much more tractable than with the real complex model. If the data set has to be determined and if the

process is time consuming and computationally expensive, then the overall usefulness of the method will depend on the use of an efficient method for selecting the fewest possible members of the data set. Design of Experiments (DOE) techniques provides the needed basis for this critical step of the methodology.

5.1 Principle of the Method

Among the methods developed for RSM construction, we adopted the one developed by Nguyen et al. [13]. The proposed method is initially derived from the method worked out by Kaymaz and McMahon [14]. It aims to improve the accuracy of results while reducing the computational cost for the purpose of applications to finite element problems where the responses of models are presented point-by-point. The main features of the proposed method are the choice of the response surface expression, the choice of the experimental design and the computation of the coefficients of the polynomial function. The response surface is built in the standardized space.

For the first iteration, a linear response surface (LRS) is chosen; the number of coefficients to be determined is $(n + 1)$. For the following iterations, a quadratic response surface with cross terms is considered (QRS). The number of coefficients to be determined is $(2n + 1)$, n being the number of variables.

Building of the DOE is adaptive. DOE is half star-shaped around a central point P_0 corresponding to the median (or the mean) values of the random variables for the first iteration. On each axis, only one point is generated, which is selected towards the failure surface. $(n + 1)$ points are thus generated. For the first DOE, the algebraic distance Δ_i between points generated on the axis “ i ” is chosen to be equal to a positive distance Δ_0 . For the following DOE, Δ_0 is determined from the distance Δ_0 (taken equal to 2) and as function of partial derivatives of the approximation $G_{equ}(x)$ at the central point [13]:

$$\Delta_i = \frac{\Delta_0}{\nabla G_{equ}(u)} \frac{\partial G_{equ}(u)}{\partial u_i} \quad (14)$$

u is the vector of random variables in the standardized space. In such a way, fitting points are located on the side of failure region (close to the central point) and at a distance proportional to the local sensitivity of the mechanical model with respect to the variable of interest.

The determination of the coefficients of the response surface is done by the least squares method. The vector of coefficients is obtained by resolving the linear system (15):

$$\bar{a} = \left(M_u^T W_G M_u \right)^{-1} M_u^T W_G \bar{g} \quad (15)$$

where:
$$M_u = \begin{bmatrix} 1 & u_{11} & \cdot & u_{n1} & u_{11}^2 & \cdot & u_{n1}^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & u_{1p} & \cdot & u_{np} & u_{1p}^2 & \cdot & u_{np}^2 \end{bmatrix}$$

g is the vector of values of the limit state function at the different p points of the DOE. In order to give greater weights to points closer to the limit state surface, Nguyen et al. [13] introduce a weighted regression to determine the coefficients of the response surface. W_G is the diagonal matrix of weight factors stated as:

$$W_G = \begin{bmatrix} W_{G1} & 0 & 0 \\ 0 & W_{Gk} & 0 \\ 0 & 0 & W_{Gn} \end{bmatrix} \text{ with } W_{Gk} = \exp\left(\frac{G(\bar{u}_k) - G(\bar{u}_0)}{G(\bar{u}_0)}\right) \cdot \exp\left(-\frac{D_k^2}{2}\right) \quad (16)$$

where $G(u_0)$ is the value of the limit state function at the origin of the standardized space (which is expected not to be zero) and where D_k is the distance between P^* (design point) and the k th point of the DOE.

5.2 General Algorithm of the Proposed Method [13]

Following Nguyen et al. [13] the general algorithm contains ten steps.

1. Building the first DOE.

The fitting points are generated from the initial point $\bar{u}_0$ (origin of the standard space). $(n + 1)$ points are generated, at the distance $+\Delta_0$ if the variable is unfavorable (load variable) and $-\Delta_0$ if the variable is favorable (resistance variable), with regard to the limit state function.

The choice of the type of a variable (load or resistance) is made by considering its influence on the physical model.

2. Computing the value of the limit state function at each point of the DOE:

$$g_k = G(\bar{u}_k), \quad k = \{1, \dots, n + 1\}.$$

3. Calculating the weight factors assigned to each g_k according to Eq. 16,

4. Fitting the LRS with Eq. 15. Computing the first reliability index β and the corresponding design point P^* ,

5. Building the complementary DOE from the new central point P_N :

$$\bar{u}_N = \bar{u}_0 + (\bar{u}_D - \bar{u}_0) \frac{G(\bar{u}_0)}{G(\bar{u}_0) - G(\bar{u}_D)} \quad (17)$$

where $G(\bar{u}_D)$ is the value of the limit state function at the point P^* . All the points of the previous DOE are preserved for the updated DOE. The point P^* is incorporated to the new DOE if it is close enough to P_N . The complementary points of the updated DOE are generated. The total number of points of the DOE is $(2n + 1)$.

6. Calculating the values of the limit state function at new points of the DOE:

$$g_k = G(\bar{u}_k), k = \{n + 2, \dots, (2n + 1)\}.$$

7. Calculating weight factors according to Eq. 16.
8. Determining the coefficients of the QRS according to Eq. 15.
9. Computing the reliability index β and the design point P^* .
10. Checking out the convergence criteria on two successive values of β and two successive points P^* .

These criteria of nearness between two successive design points ensure that the last two response surfaces suitably reproduce the behavior of the exact limit state function in the vicinity of the failure surface. If the criteria are fulfilled, the procedure stops.

Steps 8–11 are further repeated until convergence is reached.

6 Performance Functions and Limit States

Two performance functions are defined:

One relates to the counterslope and corresponds to a serviceability limit service (SLS), by relating the sewer's hydraulic performance to the local counterslope. The presence of too high counterslopes harms the flow of the effluents and facilitates the clogging of the pipes by the sedimentation of the suspended particles. The corresponding limit state function is:

$$G_{CS} = CS_R - CS_S \quad (18)$$

where CS_R is the maximum acceptable counterslope and CS_S is the maximum simulated counterslope.

The other performance function relates to the stresses and defines the equivalent of an ultimate limit state (ULS), by relating the cracking state (for a concrete pipe) to the bending stresses:

$$G_{\sigma} = \sigma_R - \sigma_S \quad (19)$$

where σ_R is the yield tension stress of the concrete and σ_S is the maximum bending stress.

We assume that CS_S as well as σ_S follow Lognormal distributions, whose means and standard deviations are obtained from coupling RSM and finite element method, and that CS_R as σ_R also follow Lognormal distributions of means and standard deviations respectively of $4\% \pm 0.8\%$, and 2 ± 0.3 MPa.

It should be noted that there does not exist any regulation limiting the counterslope, that is why we have chosen a 4% arbitrary limit based on the design considerations: trespassing such a value leads to damages or dismantling of the joints and to defects of sealing. The $\pm 0.8\%$ can be seen as a model uncertainty

on this SLS criterion. The target values of the reliability index to be reached are respectively 3.8 for the ULS and 1.5 for the SLS with a service life of 50 years [1]. In following, we will consider two reliability indexes β_{ULS} and β_{SLS} .

7 Case Study

7.1 Characteristics of the Sewer's Section

Consider a reinforced concrete material for the pipe ($E_p = 25,000$ MPa) with an exterior diameter $D_{\text{ext}} = 1$ m, a thickness of 0.08 m, made of a set of 20 pipes with unit length ($L = 3$ m). It rests on a soil whose Winkler coefficient is 10 kN/m^3 and shear coefficient of 10 kN/m . This type of configuration corresponds to a pipe–soil ratio r_p of 0.678. It must be noted that the pipe geometry intervenes as well in the expression of inertia I as in that of the Winkler coefficients. The pipes are subjected to a deterministic uniform loading of 50 kN/m and to the earthquake of El-Asnam (Algeria 1980).

One defines λ the ratio between the length of correlation l_c and the pipe length L . In the following sections are analyzed:

- the effect of the length of correlation on the response of the system,
- and for an unfavorable length of correlation ($\lambda \neq 1$), the effect of the relative soil-pipe stiffness,
- and the effect of the relative joint-pipe stiffness.

The values of the reliability β indexes (ULS or SLS) obtained must be considered rather comparatively that in an absolute way because of:

- the arbitrary character of the values retained for CS_R as σ_R ,
- the type of the probabilistic distribution,
- model uncertainties on soil characteristics (k_w , k_s).

7.2 Effect of the Correlation Length

The analysis is carried out for lengths of correlation ranging from 0.03 to 1,500 m (otherwise $10^{-3} < \lambda < 5 \times 10^2$). Two types of joints are considered: rigid ($r_{\text{joint}} = 10^{+5}$) and flexible ($r_{\text{joint}} = 10^{-5}$).

Figures 4 and 5 show the ULS and SLS reliability indexes. One notes that: It is shown the existence of a critical length of correlation and thus a critical value of λ which causes the most unfavorable effects in the structure. The evolution of the reliability is not monotonous. The existence of such a limit value is one of the invariants of the problems of (heterogeneous) soil-structure interaction [10].

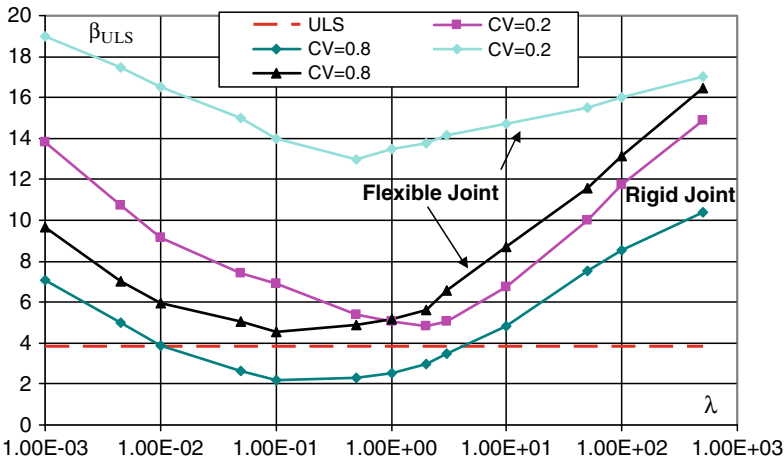


Fig. 4 Effect of correlation length on ultimate limit state reliability index for two values of the coefficient of variation of k_w and two types of joints

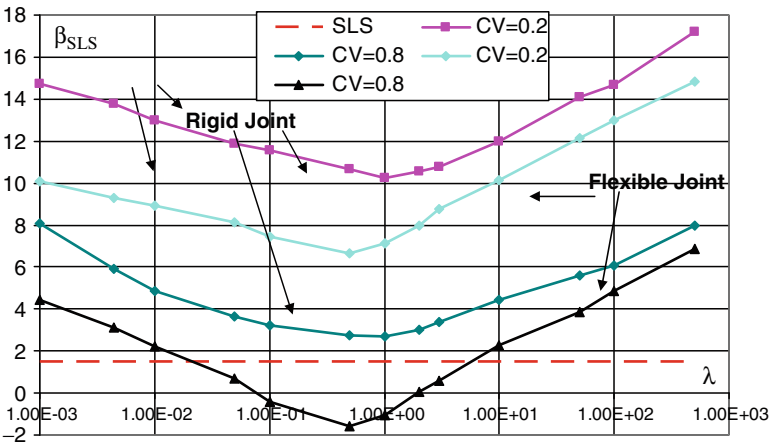


Fig. 5 Effect of correlation length on service limit state reliability index for two values of the coefficient of variation of k_w and two types of joints

Reliability indexes grow when λ tends towards zero or infinity: in these two situations the soil tends to be homogeneous at the scale of the analysis, either because its variations are filtered very fast by the pipe, or because its variations are very slow and induce almost uniform values between close elements.

We had seen that the rigid joints and for a high coefficient of variation of k_w do not satisfy the ULS for λ comprised between 0.01 and 7, while the flexible joints provide results much better. On the other hand if we consider the SLS, the flexible joints provide results a little worse than the rigid joints. One thus notes at this level a conflict in the choice of the “good pipe length” to satisfy the ULS as well as the SLS.

7.3 Effect of the Joint-Pipe Stiffness Ratio

The effect of joints on the behavior of the system was analyzed by varying the joint-pipe ratio r_{joint} from 10^{-5} to 10^{+5} . Figures 6 and 7 show the evolution of the reliability indexes for different coefficients of variation of Winkler coefficient.

It is interesting to underline the strong influence of the joints stiffness since the rigid joints are penalizing with the ULS, by the fact that they necessarily induce more significant stresses. On the other hand, the flexible joints are penalizing with the SLS. One has to choose an intermediate joint stiffness in order to carry out the best compromise, since reliability is completely assured as well for the ULS as the SLS. Finally, one can also note that for a given value of r_{joint} , as well as ULS index reliability than SLS index reliability decrease with the increase of k_w coefficients of variation.

7.4 Effect of the Soil-Pipe Stiffness Ratio

By maintaining the same set of pipes and soil parameters ($\lambda = 1$, $r_{ks} = 1$), we vary the soil-pipe stiffness ratio r_p in the range [0.4–1.7] which corresponds to a Winkler coefficient in the range [100 MPa to 100 kPa] for four values of the coefficient of variation ($\text{CoV}_{k_w} = 0.1, 0.2, 0.4$ and 0.8).

While taking here a length of correlation l_c close the length of the pipe ($\lambda \neq 1$), we refer to the situation in the vicinity of the most unfavorable geotechnical conditions. Figures 8 and 9 present the values of the reliability indexes. Whatever the type of

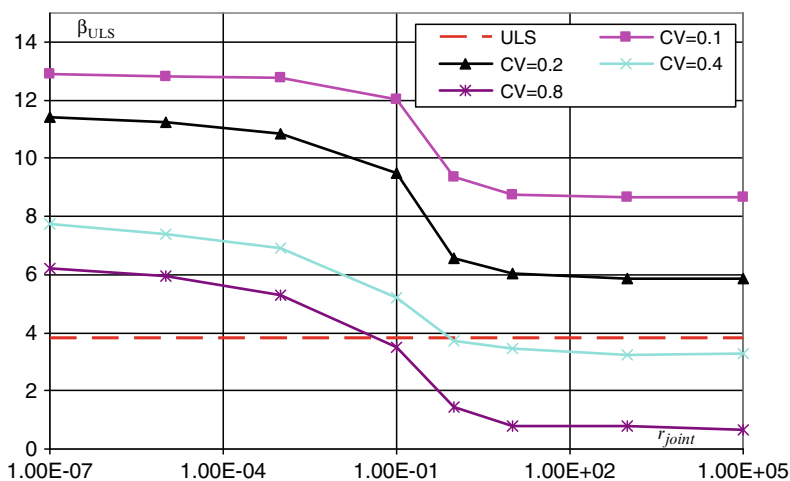


Fig. 6 Effect of joint-pipe stiffness ratio on ultimate limit state reliability index for four different values of the coefficient of variation of k_w

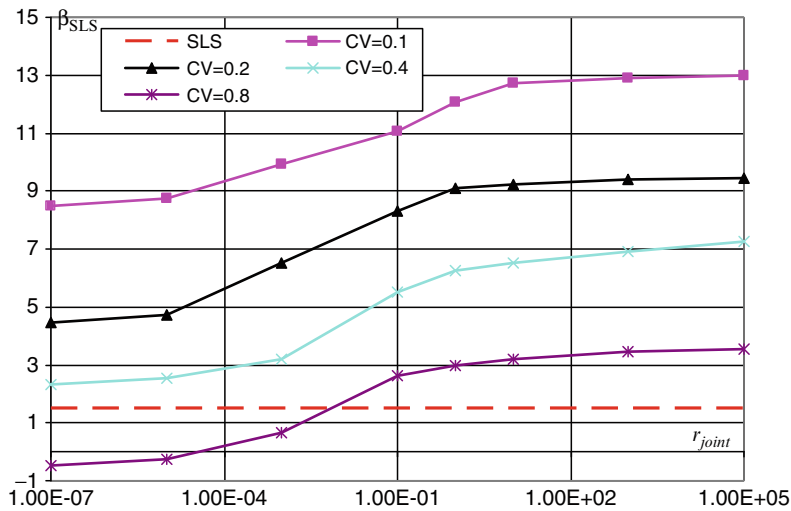


Fig. 7 Effect of joint-pipe stiffness ratio on service limit state reliability index for four different values of the coefficient of variation of k_w

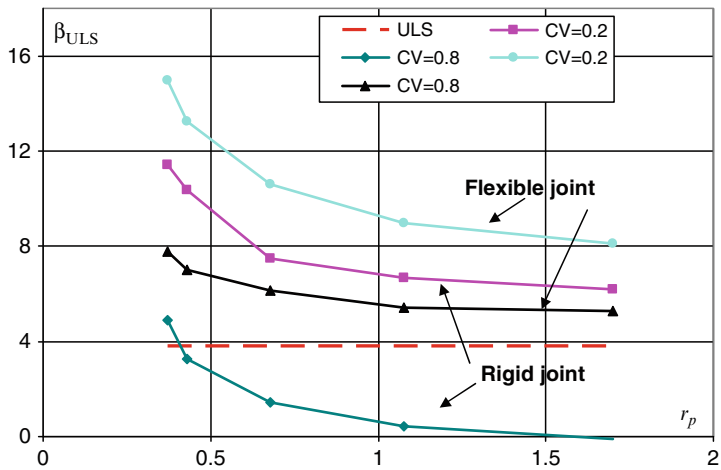


Fig. 8 Effect of soil-pipe stiffness ratio on ultimate state reliability index for four different values of the coefficient of variation of k_w

joint, reliability is lower for high soil-pipe stiffness ratio, situation corresponding to sections of very stiff pipes or to low soil stiffness. In the presence of rigid joints (Fig. 8) and high coefficient of variation of the soil's parameter, the Eurocode β_{ULT} is not respected (lower than 3.8). One notes that increasing the coefficient of variation induces reduction of the reliability of the pipe section. One can see it is better to

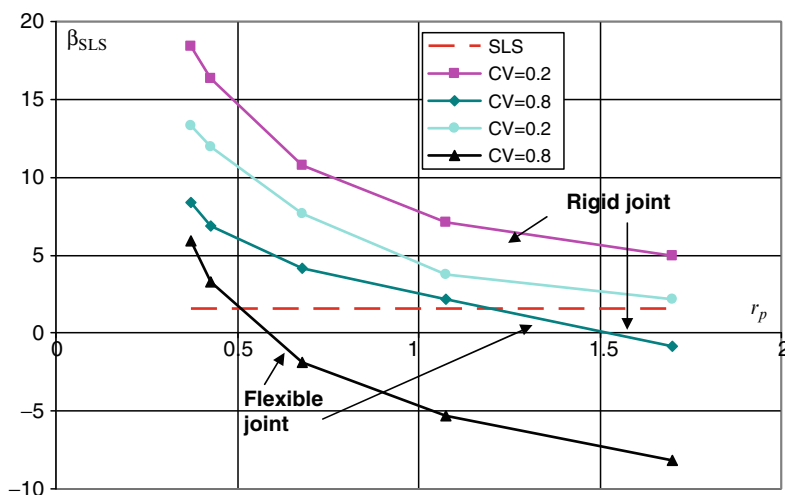


Fig. 9 Effect of soil–pipe stiffness ratio on service limit state reliability index for four different values of the coefficient of variation of k_w

have flexible joints than rigid joints, better is worth to privilege a lower r_p to ensure “a good” reliability of the network.

In terms of counterslopes (Fig. 9), it is better to have rigid joints than flexible joints. To ensure the reliability of the sewer’s section using rigid joints imposes that the ground is rather homogeneous.

One can also note that it is more interesting to be in the presence of a relatively homogeneous soil and whose mechanical characteristics are poor than in the presence of a better ground but whose heterogeneity is more pronounced.

8 Conclusions

Several conclusions are drawn:

- Soil heterogeneity induces effects (differential settlements, bending moments, stresses and possible cracking) that cannot be predicted if homogeneity is assumed.
- The magnitude of the induced stresses depends mainly on four factors:
 - The magnitude of the soil variability (i.e. its coefficient of variation)
 - A soil–structure stiffness ratio
 - A structure–joint stiffness ratio (relative flexibility)
 - A soil–structure length ratio, which combines the soil fluctuation scale and a structural characteristic length (buried pipe length)

A worst value, corresponding to the value leading (from a statistical point of view) to the (statistically) largest effects in the structure, can be found. The principal benefit of such an approach is to provide some new approaches for better considering phenomena such as the geometrical irregularities in the longitudinal profile during the control of soil compaction of sewer trench filling. This kind of approach can also give experts new tools for better calibration of safety in soil-structure interaction problems, when the soil variability is an influential parameter.

References

1. Eurocode 7, Geotechnical design, Part 1: general rules. Setra (1997)
2. Manolis, G.D., Beskos, D.E.: Underground structures and lifelines. In: Beskos, D.E., Anagnostopoulos, S.A. (eds.) *Computer Analysis and Design of Earthquake Resistant Structures*, pp. 775–837. Computational Mechanics Publications, Southampton (1997)
3. Pasternak, P.L.: On a new method of analysis of an elastic foundation by means of two foundation constants (in Russian). Gosudarstvennoe Izdatelstvo Literaturi po Stroitelstvu Arkhitekture, Moscow, USSR (1954)
4. Elachachi, S.M., Bensafi, M., Hamane, M., Nedjar, D., Breysse, D.: Longitudinal flexural behavior of a pipe buried in an heterogeneous embankment. *Geotech. Fr. Rev.* 108, 17–29 (2004)
5. Morfidis, K., Avramidis, I.E.: Formulation of a generalized beam element on a two-parameter elastic foundation with semi-rigid connections and rigid offsets. *Comput. Struct.* 80, 1919–1934 (2002)
6. Elachachi, S.M., Breysse, D., Laurent, H.: Longitudinal variability of soils and structural response of sewer networks. *Comput. Geotech.* 31, 625–641 (2004)
7. Buco, J., Emeriault, F., Le Gauffre P., Kastner R.: Statistical and 3D numerical identification of pipe and bedding characteristics responsible for longitudinal behavior of buried pipe, pipelines 2006. The pipeline division specialty conference, USA, July 30–August 2, 2006
8. Duncan, J.M.: Factors of safety and reliability in geotechnical engineering. *J. Geot. Geoenviron. Eng. ASCE* 126(4), 307–314 (2000)
9. Kovarik, J.B.: A propos des valeurs caractéristiques des propriétés des sols. In *Journées ENPC de présentation des Eurocodes* (1996)
10. Breysse, D., Niandou, H., Elachachi, S.M., Houy, L.: Generic approach of soil-structure interaction considering the effects of soil heterogeneity. *Geotechnique* LV(2), 143–150 (2005)
11. VanMarcke, E.: *Random Fields: Analysis and Synthesis*. M.I.T. Press, Cambridge, MA (1983)
12. Nedjar, D., Hamane, M., Bensafi, M., Elachachi, S.M., Breysse, D.: Seismic response analysis of pipes by a probabilistic approach. *Soil Dyn. Earthquake Eng.* 27(2), 111–115 (2007)
13. Nguyen, X.S., Sellier, A., Duprat, F., Pons, G.: Adaptive response surface method based on a double weighted regression technique. *Probab. Eng. Mech.* 24(2), 135–143 (2009)
14. Kaymaz, I., McMahon, C.A.: A response surface method based on weighted regression for structural reliability analysis. *Probab. Eng. Mech.* 20, 1–7 (2004)

An Efficient First-Order Scheme for Reliability Based Optimization of Stochastic Systems

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Abstract This contribution presents a novel approach for solving reliability-based optimization (RBO) problems. In particular, the reliability-based optimization of structural systems under stochastic loading is considered. The approach is based on a descent-feasible direction and a line search strategy, allowing to explore the space of the design variables most efficiently. The proposed method is monotonically convergent, that is, it generates a sequence of steadily improved feasible designs. An efficient simulation technique is used for the purpose of evaluating the reliability constraints. On the other hand, the sensitivity of the reliability constraints with respect to the design variables is estimated by an approach based on the local behavior of the performance functions that define the failure domains. Numerical results show that only a moderate number of reliability estimates has to be performed during the design process. A numerical example showing the efficiency and effectiveness of the approach reported herein is presented.

1 Introduction

Structural optimization is concerned with achieving an optimal design while satisfying certain constraints. In this regard, the optimal design can be defined as the best feasible design according to a preselected quantitative measure of effectiveness. In most structural engineering applications, response predictions are based on structural models whose parameters are uncertain. This is due to a lack of information about the value of system parameters external to the structure, such as environmental loads (wind loading, water wave excitation, traffic loading, earthquake

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excitation, etc.) or internal such as material properties, construction defects, system behavior, etc. Under uncertain conditions the field of reliability-based optimization provides a realistic and rational framework for structural optimization which explicitly accounts for the uncertainties. Standard reliability based optimization formulations include problems such as: minimization of construction costs under probabilistic constraints; minimization of failure probability for a fixed structural cost; maximization of overall utility of structures; minimization of lifetime costs of facilities; design for optimal inspection and maintenance planning; etc [9]. Reliability-based optimization requires advanced and efficient tools for structural modeling, reliability analysis and mathematical programming. This is especially important when dealing with structural systems subject to stochastic excitation.

Modeling and analysis techniques of mechanical systems based on local approximations are well established and sufficiently well documented in the literature [4, 39]. Techniques such as finite and boundary element methods have been extensively used in structural analysis and design. On the other hand, several tools for assessing structural reliability have lately experienced a substantial development providing solution to a number of complex problems. Simulation methods such as Subset Simulation, Line Sampling, Importance Sampling, and the Domain Decomposition Method have played a major role in assessing structural reliability of realistic and challenging engineering problems involving large structural models [3, 5, 15–17, 19, 27, 33]. In the field of reliability-based optimization several procedures have been recently developed allowing the solution of quite demanding problems. Such procedures are usually based on a combination of approximation concepts with standard deterministic optimization techniques [1, 7, 8, 10, 12, 13, 18, 23, 25, 26], or stochastic search algorithms [20, 35].

The approximate models, i.e. meta-models, are usually simple functions of the design variables. Thus, the optimal design of the approximate model can be easily evaluated and serves as an approximation to the original reliability-based optimization problem. In other words, the optimizer of the meta-model is used as an estimate of the real (exact) optimizer. Methods such as response surfaces, neural networks, Kriging models, and local and global approximation techniques have been proposed as meta-modeling techniques in the area of design optimization. In this context, recent developments of efficient and robust sensitivity analysis techniques are closely related to the construction of meta-models for complex structural systems [2, 24, 30, 36, 38].

Stochastic search algorithms have also proved to be useful tools for solving challenging optimization problems. In these approaches the values of the random functions are used directly as inputs to the optimization algorithm. The algorithms used in these cases are generally direct search schemes which only use the values of random functions to be optimized as inputs. Thus, these approaches represent the most direct usage of information obtained by simulation techniques. In the presence of uncertainty several types of direct search methods can be identified. They include stochastic approximation methods, evolutionary algorithms, pattern search methods, and evolutionary algorithms [25, 30, 34]. For a thorough review of the previous and other recent advances in the context of optimization problems considering uncertainties it is referred to, e.g. [31, 32].

The use of the above optimization approaches has been found useful in a number of structural optimization applications. However, the application of reliability-based optimization to stochastic dynamical systems remains somewhat limited. For example, meta-modeling techniques are not well suited to large scale optimization problems when the number of design variables is large. This is specially prohibitive when considering large scale simulation models. Also, optimization of the meta-model will usually be only a first approximation. That is, the meta-model must be repeatedly applied in order to get a closer optimum. On the other hand, most of the available optimization schemes dealing with stochastic dynamical systems do not possess proven convergence properties. Therefore, there is still much room for further developments, considering for example more design variables, large-scale mechanical models, provable robustness properties in terms of convergence, etc. In this work an optimization scheme based on feasible-descent directions and a line search strategy is presented. The approach has monotonic convergence properties, that is, the process may be stopped at any stage still leading to feasible designs better than the initial estimate. In addition, the proposed scheme can handle complex structural models in a very efficient manner.

The structure of the paper is as follows. First, the mathematical formulation for the reliability-based optimization problem is presented. The proposed optimization scheme is then considered. Next, several implementation issues such as: reliability assessment; gradient estimation of failure probability functions; line search implementation; and reliability estimates variability are discussed in detail. Finally, an application problem is presented to illustrate the performance of the proposed methodology.

2 Reliability-Based Optimization Problem

2.1 Formulation

Consider the following structural optimization problem

$$\text{Min } C(\{x\})$$

subject to

$$\begin{aligned} h_i(\{x\}) &\leq 0, \quad i = 1, \dots, n \\ g_i(\{x\}) &\leq 0, \quad i = 1, \dots, m \\ s_i(\{x\}) &= \{a_i\}^T \{x\} - b_i \leq 0, \quad i = 1, \dots, l \end{aligned} \quad (1)$$

where $\{x\}, x_i, i = 1, \dots, n_d$ is the vector of design variables, $C(\{x\})$ is the objective function, $h_i(\{x\}) \leq 0, i = 1, \dots, n$ are the reliability constraints, $g_i(\{x\}) \leq 0, i = 1, \dots, m$ are the deterministic nonlinear constraints, and

$s_i(\{x\}) \leq 0$, $i = 1, \dots, l$ are the deterministic linear constraints. The deterministic constraints are related to design requirements such as structural weight, geometric conditions, material cost components, etc. The side constraints

$$\{x\} \in X, \quad x_i \in X_i = \{x_i | x_i^l \leq x_i \leq x_i^u\}, \quad i = 1, \dots, n_d \quad (2)$$

are included in the definition of the linear deterministic constraints $s_i(\cdot)$. The reliability constraints are written in terms of the failure probability functions as

$$h_i(\{x\}) = P_{F_i}(\{x\}) - P_{F_i}^* \leq 0, \quad i = 1, \dots, n \quad (3)$$

where $P_{F_i}(x)$ is the failure probability function for the failure event F_i evaluated at the design $\{x\}$, and $P_{F_i}^*$ is the target failure probability for the i th failure event. The failure probability function $P_{F_i}(\{x\})$ evaluated at the design $\{x\}$ can be written in terms of the probability integral

$$P_{F_i}(\{x\}) = \int_{\Omega_{F_i}} f(\{z\}/\{x\}) d\{z\} \quad (4)$$

where $\{z\} \in \Omega_{\{z\}} \subset R^{n_u}$ is the vector of uncertain variables involved in the problem, $f(\{z\}/\{x\})$ is the probability density function of $\{z\}$ conditioned on $\{x\}$, and Ω_{F_i} is the failure domain of failure event F_i in the $\Omega_{\{z\}}$ space. The failure domain Ω_{F_i} for a given design $\{x\}$ is defined in terms of the performance function κ_i as $\kappa_i(\{x\}, \{z\}) \leq 0$, that is $\Omega_{F_i} = \{\{z\} \mid \kappa_i(\{x\}, \{z\}) \leq 0\}$. Recall that $\{z\}$ is the vector of random variables that describes all uncertainties involved in the system (model and loading parameters). That is, the components of the vector $\{z\}$ represent the uncertain structural parameters and the random variables used in the characterization of the stochastic excitation. Therefore, the failure probability functions $P_{F_i}(\{x\})$, $i = 1, \dots, n$ account for the uncertainty in the system parameters as well as the uncertainties in the excitation.

2.2 Application to Dynamical Systems

For systems under stochastic excitation the probability that design conditions are satisfied within a particular reference period provides a useful reliability measure. Such measure is referred as the first excursion probability. In this case the failure events F_i , $i = 1, \dots, n$ are defined as

$$F_i(\{x\}, \{z\}) = D_i(\{x\}, \{z\}) > 1 \quad (5)$$

where

$$D_i(\{x\}, \{z\}) = \max_{j=1, \dots, n_j} \max_{t \in [0, T]} \frac{|r_j^i(t, \{x\}, \{z\})|}{r_j^{i*}} \quad (6)$$

is the normalized demand, $[0, T]$ is the time interval, $r_j^i(t, \{x\}, \{z\})$, $j = 1, \dots, n_j$ are the response functions associated with the failure event i , and r_j^{i*} is the corresponding critical threshold level. In this context the quotient $|r_j^i(t, \{x\}, \{z\})| / r_j^{i*}$ is interpreted as a demand to capacity ratio, as it compares the value of the response $r_j^i(t, \{x\}, \{z\})$ with the maximum allowable value of this response r_j^{i*} . The response functions $r_j^i(t, \{x\}, \{z\})$, $i = 1, \dots, n$, $j = 1, \dots, n_j$ are obtained from the solution of the equation of motion that characterizes the structural model. With the previous definition of normalized demand, the performance function can be written as

$$\kappa_i(\{x\}, \{z\}) = 1 - D_i(\{x\}, \{z\}) \quad (7)$$

and the corresponding first excursion probability can be expressed as the multidimensional integral

$$P_{F_i}(\{x\}) = \int_{\kappa_i(\{x\}, \{z\}) \leq 0} f(\{z\}/\{x\}) d\{z\} \quad (8)$$

3 Proposed Approach

3.1 General Description

A first-order optimization scheme based on feasible directions has been selected in the present implementation for its simplicity and robustness [11]. Starting from an interior point in the design space the first step of the approach is to determine an active design since the optimum solution for most practical applications occurs on the boundary of the design space. An active design is a point on the design space boundary, i.e. one or more inequality constraints are active. Next, a one-dimensional search is carried out along a given direction in order to obtain a new design (better than the previous one) and the corresponding active constraints. The process continues until convergence is achieved. By construction the method generates a sequence of steadily improved feasible designs. In what follows, the basic concepts involved in the proposed optimization scheme such as descent directions, feasible directions, active set of constraints, and one-dimensional search are presented. It is emphasized that the proposed scheme can be extended in a straightforward manner to incorporate more advanced first-order optimization algorithms.

3.2 Descent and Feasible Directions

The objective function $C(\cdot)$, the reliability constraint functions $h_i(\cdot)$, $i = 1, \dots, n$, and the deterministic constraint functions $g_i(\cdot)$, $i = 1, \dots, m$ are assumed to

have continuous derivatives. A descent direction $\{d\}$ at the design $\{x\}$ verifies the condition $\nabla C^T \{d\} < 0$, which ensures that for sufficiently small $\varepsilon > 0$ the inequality

$$C(\{x\} + \varepsilon\{d\}) < C(\{x\}) \quad (9)$$

should hold. Next, a feasible design $\{x\}$, that is $\{x\} \in \Omega$, where Ω denotes the feasible region is considered. The feasible region is defined by the collection of designs that satisfy

$$\begin{aligned} \Omega = \{ \{x\} \mid h_i(\{x\}) \leq 0, \quad i = 1, \dots, n, \quad g_i(\{x\}) \leq 0, \quad i = 1, \dots, m, \\ s_i(\{x\}) \leq 0, \quad i = 1, \dots, l \} \end{aligned} \quad (10)$$

A feasible direction $\{d\}$ at the design $\{x\}$ is such that there is a $\bar{\varepsilon} > 0$ such that $\{x\} + \varepsilon\{d\} \in \Omega$ for all $\varepsilon, 0 \leq \varepsilon \leq \bar{\varepsilon}$. At the feasible design $\{x\}$ let one or more constraints be active. In this context the active set of constraints J is defined as

$$\begin{aligned} J = \{ j : h_j(\{x\}) = 0, j = 1, \dots, n \} \cup \{ j : g_j(\{x\}) = 0, j = 1, \dots, m \} \\ \cup \{ j : s_j(\{x\}) = 0, j = 1, \dots, l \} \end{aligned} \quad (11)$$

A feasible direction $\{d\}$ satisfies the condition

$$\nabla h_i(\{x\})^T \{d\} < 0, \quad \nabla g_i(\{x\})^T \{d\} < 0, \quad \text{and} \quad \nabla s_i(\{x\})^T \{d\} < 0 \quad (12)$$

for each $i \in J$ which ensure that for sufficiently small $\varepsilon > 0$ the design $\{x\} + \varepsilon\{d\}$ will be feasible. In the previous definition of feasible direction it has been assumed that the design $\{x\}$ is a regular point, i.e. the gradient vectors $\nabla h_i(\{x\})$, $\nabla g_i(\{x\})$, $\nabla s_i(\{x\})$ $i \in J$ are linearly independent.

3.3 Direction Evaluation

3.3.1 Interior Design

If the current design $\{x_k\}$ is an interior point in the feasible design space Ω , then the active set is empty and the steepest descent direction $-\nabla C(\{x\})$ is the obvious choice for $\{d\}$. Proceeding along, there is a high chance to hit the boundary of the feasible design space as most realistic engineering problems have the minimum on the boundary. Therefore, it can be assumed in what follows that some constraints (probabilistic or deterministic) are active at the next design $\{x_{k+1}\}$.

3.3.2 Active Non-Linear Constraints

To determine a descent feasible direction from a design where some of the active constraints are non-linear the following optimization problem is formulated

$$\text{Min } \alpha$$

subject to

$$\begin{aligned} \nabla C(\{x_k\})^T \{d\} &\leq \alpha, \quad \nabla h_j(\{x_k\})^T \{d\} \leq \alpha, \quad j \in J \\ \nabla g_j(\{x_k\})^T \{d\} &\leq \alpha, \quad j \in J, \quad \nabla s_j(\{x_k\})^T \{d\} \leq \alpha, \quad j \in J \\ -1 &\leq d_i \leq 1, \quad i = 1, \dots, n_d \end{aligned} \quad (13)$$

where the optimization variables are given by the component of the vector $\{d\}$ and the scalar α . The constraints on the magnitude of d_i have been imposed to ensure a bounded solution. It is clear that a negative α means that the vector $\{d\}$ is a descent-feasible direction at the current design. A schematic illustration of the descent feasible direction concept is shown in Fig. 1. The optimization problem (13) can be rewritten as a linear programming problem (LP) by defining the variables $v_i = d_i + 1, i = 1, \dots, n_d$ and $\beta = -\alpha$. Substituting these variables in (13), the optimization problem can be arranged as

$$\text{Minimize } \{q\}^T \{y\}$$

such that

$$[A]\{y\} \leq \{c\}, \quad \{y\} \geq 0 \quad (14)$$

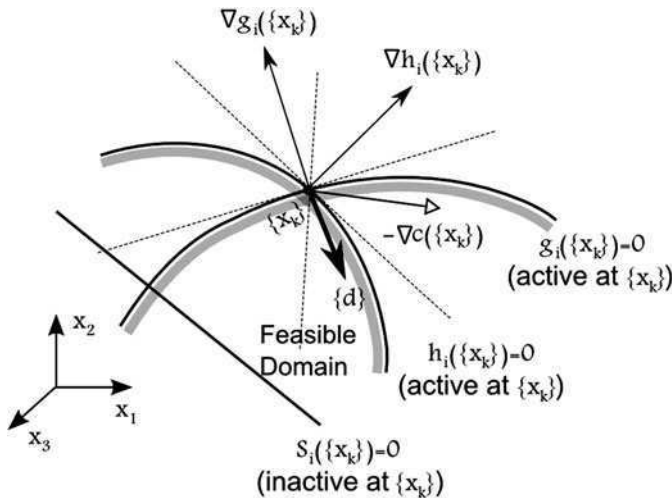


Fig. 1 Descent feasible direction

with

$$\begin{aligned} \{y\}^T &= (\{v\}^T, \beta), \quad \{q\}^T = (0, \dots, 0, -1) \\ \{c\}^T &= (c_0, \dots, c_1^j, \dots, c_2^j, \dots, c_3^j, \dots, 2, \dots, 2) \\ &\qquad\qquad\qquad j \in J \end{aligned} \quad (15)$$

$$A = \begin{bmatrix} \nabla C^T(\{x\}) & 1 \\ \cdot & 1 \\ \cdot & 1 \\ \nabla h_j^T(\{x\}) & 1 \\ \cdot & 1 \\ \cdot & 1 \\ \nabla g_j^T(\{x\}) & 1 \\ \cdot & 1 \\ \cdot & 1 \\ \nabla s_j^T(\{x\}) & 1 \\ \cdot & 1 \\ \cdot & 1 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & 1 & 0 \end{bmatrix} \quad (16)$$

where

$$\begin{aligned} \nabla C^T(\{x_k\}) \begin{Bmatrix} 1 \\ \vdots \\ 1 \end{Bmatrix} &= c_0, \quad \nabla h_j^T(\{x_k\}) \begin{Bmatrix} 1 \\ \vdots \\ 1 \end{Bmatrix} = c_1^j \\ \nabla g_j^T(\{x_k\}) \begin{Bmatrix} 1 \\ \vdots \\ 1 \end{Bmatrix} &= c_2^j, \quad \nabla s_j^T(\{x_k\}) \begin{Bmatrix} 1 \\ \vdots \\ 1 \end{Bmatrix} = c_3^j \end{aligned} \quad (17)$$

A standard simplex method is used to solve the above problem for $d_i^* = v_i^* - 1$ and β^* . The gradients ∇C , ∇h_i and ∇g_i used in the previous LP problem are normalized as unit vector by dividing by their lengths. If $\beta^* > 0$ ($\alpha^* < 0$) means that the vector $\{d\}^*$ is a descent-feasible direction. On the other hand, if $\beta^* = 0$ means that no descent-feasible direction exists, and therefore convergence is achieved [6]. As previously pointed out, if $\beta^* > 0$, a descent-feasible direction exists and line search (one dimensional search) has to be performed (see Sect. 3.4).

3.3.3 Active Linear Constraints

If at the current design $\{x_k\}$ the active set of constraints are only linear, the direction vector $\{d\}$ can be obtained directly by projecting the steepest descent direction $-\nabla C(\{x_k\})$ onto the tangent plane defined by the active linear constraints [29]. The tangent plane is characterized by

$$[B]\{p\} = \{0\} \quad (18)$$

where the matrix $[B]$ consists of rows of the gradient vectors obtained from the active linear constraints, i.e.

$$[B]^T = \{\{a_1\}^T, \{a_2\}^T, \dots, \{a_i\}^T, \dots\}, \quad i \in J \quad (19)$$

The descent-feasible direction $\{d\}$ is given by

$$\{d\} = -\nabla C(\{x_k\}) - [B]^T \{\delta\} \quad (20)$$

where $\{\delta\}$ is the solution of the equation

$$[B][B]^T \{\delta\} = -[B]\nabla C(\{x_k\}) \quad (21)$$

where it has been assumed that the current design is regular, that is, the rows of $[B]$ are linearly independent. If there is a dependency, the redundant constraints must be deleted [22]. Computationally, the direction vector $\{d\}$ is obtained by combining Eqs. 20 and 21 which gives

$$\{d\} = [P](-\nabla C(\{x_k\})), \quad [P] = I - [B]^T ([B][B]^T)^{-1} [B] \quad (22)$$

where $[P]$ is the projection matrix. The product $[P](-\nabla C(\{x_k\}))$ projects the steepest descent direction $-\nabla C(\{x_k\})$ onto the tangent plane, as shown in Fig. 2.

3.4 One Dimensional Search

Once a descent-feasible direction has been obtained, a one dimensional search is carried out. This is a constrained one-dimensional search and it can be expressed as

$$\text{Min } C^d(\alpha) = C(\{x_k\} + \alpha\{d\})$$

subject to

$$\begin{aligned} h_i^d(\alpha) &= h_i(\{x_k\} + \alpha\{d\}) \leq 0, \quad i = 1, \dots, n \\ g_i^d(\alpha) &= g_i(\{x_k\} + \alpha\{d\}) \leq 0, \quad i = 1, \dots, m \\ s_i^d(\alpha) &= s_i(\{x_k\} + \alpha\{d\}) \leq 0, \quad i = 1, \dots, l \end{aligned} \quad (23)$$

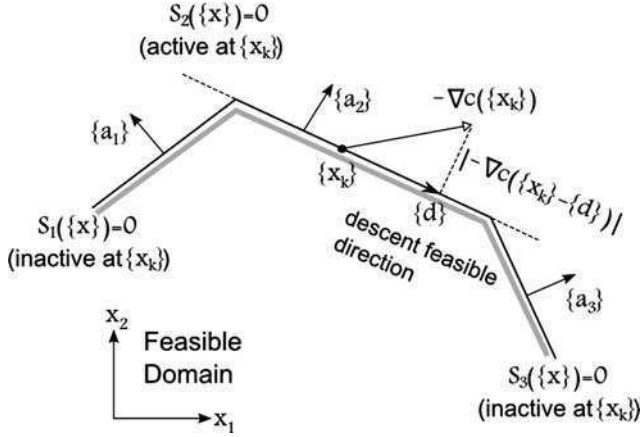


Fig. 2 Projection of the steepest descent direction onto the tangent plane: $\{d\}=[P](-\nabla C(\{x_k\}))$

First, the step $\bar{\alpha}$ to the nearest intersecting boundary (active deterministic or probabilistic constraint) is determined. The actual implementation of this step is discussed in Sect. 4.3. Having determined the step $\bar{\alpha}$, the optimal step α_k^* is evaluated as follows. If the slope of the objective function $C^d(\bar{\alpha})$, that is

$$\left. \frac{dC^d(\{x\})}{d\alpha} \right|_{\alpha=\bar{\alpha}} = \nabla C^T(\{x_k\} + \bar{\alpha}\{d\}) \{d\} \quad (24)$$

is negative, then the optimal step is given by $\alpha_k^* = \bar{\alpha}$, and the new point in the design space is

$$\{x_{k+1}\} = \{x_k\} + \alpha_k^* \{d\} \quad (25)$$

On the other hand, if the slope of the objective function $C^d(\bar{\alpha})$ is positive, the minimum value of C is obtained in the interval $[0, \bar{\alpha}]$. The optimum value of α can be determined for example by using a bisection strategy [11].

3.5 Algorithm Description

The proposed algorithm is summarized as follows.

1. Initial design

Determine an interior starting design $\{x_k\}$, $k = 0$.

In many practical applications a feasible design can be obtained from physical considerations or engineering criteria. If this is not the case several techniques are available for finding an initial feasible design [6, 14].

2. Initial active set of constraints

Perform a one dimensional search in the feasible region along the steepest descent direction $-\nabla C(\{x_k\})$, $k = 0$ to determine an active set from

$$J = \{j : -\epsilon \leq h_j(\{x_k\}) \leq 0, \quad j = 1, \dots, n\} \cup \\ \{j : -\epsilon \leq g_j(\{x_k\}) \leq 0, \quad j = 1, \dots, m\} \cup \\ \{j : -\epsilon \leq s_j(\{x_k\}) \leq 0, \quad j = 1, \dots, l\} \quad (26)$$

where the user-defined scalar ϵ or constraint thickness parameter allows constraints near the limit to be included in the active set.

3. Determine the direction vector $\{d\}$

- a. If the active set is defined only by linear constraints, form the matrix $[B]$ (Eq. 19), and solve Eq. 21 for $\{\delta\}$ and evaluate $\{d\}$ from Eq. 20.
- b. If the active set contains nonlinear constraints, solve the linear programming problem (14) to find $\{d^*\} = \{v^*\} - \{1\}$, where the components of the vector $\{1\}$ are equal to one.

4. One dimensional search

Perform line search and determine the optimum step size α_k^* . Update the current design $\{x_k\}$ as $\{x_{k+1}\} = \{x_k\} + \alpha_k^* \{d\}$.

Determine the active set of constraints and return to step (3).

5. Check for convergence

If at the current design a descent-feasible direction does not exist, the optimization scheme stops.

3.6 Algorithm Properties

The optimization strategy adopted in this work is based on a local optimization algorithm. Therefore the optimization process can converge to a local optimum. This situation may occur in structural optimization problems with, for example, non-convex objective functions, or non-convex or disjoint design spaces. Thus, the solution of the optimization process does not ensure the identification of the global optimum. That is, the selection of a particular initial solution can affect the value of the final design. In spite of this fact, local optimization schemes are valuable tools in the design process of structural systems. Attempts to find the global optimum (or optima) can be based on physical considerations of the problem or the use of global optimization algorithms. It is noted that, however, global optimization schemes require a large number of function evaluations (i.e. reliability estimations) and therefore they are impractical for the optimization of dynamical systems under stochastic excitation. In this regard the implementation of global optimization strategies is the subject of future research.

4 Implementation Issues

4.1 Reliability Assessment

The reliability constraint functions $h_i(\{x\})$, $i = 1, \dots, n$ of the stochastic optimization problem (1) are given in terms of the first excursion probability functions $P_{F_i}(\{x\})$, $i = 1, \dots, n$. Subset simulation [3] is adopted in this formulation for the purpose of estimating the corresponding failure probabilities during the design process. In the approach, the failure probabilities are expressed as a product of conditional probabilities of some chosen intermediate failure events, the evaluation of which only requires simulation of more frequent events. Therefore, a rare event simulation problem is converted into a sequence of more frequent event simulation problems. For example, the failure probability $P_{F_i}(\{x\})$ can be expressed as the product

$$P_{F_i}(\{x\}) = P(F_{i,1}(\{x\})) \prod_{k=1}^{m_F-1} P(F_{i,k+1}(\{x\})/F_{i,k}(\{x\})), \quad (27)$$

where $F_{i,m_F}(\{x\}) = P_{F_i}(\{x\})$ is the target failure event and $F_{i,m_F}(\{x\}) \subset F_{i,m_F-1}(\{x\}) \dots \subset F_{i,1}(\{x\})$ is a nested sequence of failure events. Equation 27 expresses the failure probability $P_{F_i}(\{x\})$ as a product of $P(F_{i,1}(\{x\}))$ and the conditional probabilities $P(F_{i,k+1}(\{x\})/F_{i,k}(\{x\}))$, $k = 1, \dots, m_F - 1$. It is seen that, even if $P_{F_i}(\{x\})$ is small, by choosing m_F and $F_{i,k}(\{x\})$, $k = 1, \dots, m_F - 1$ appropriately, the conditional probabilities can still be made sufficiently large, and therefore they can be evaluated efficiently by simulation because the failure events are more frequent. The intermediate failure events are chosen adaptively using information from simulated samples so that they correspond to some specified values of conditional failure probabilities. For details of this simulation procedure from the theoretical and numerical viewpoint it is referred to, e.g. [3].

4.2 Gradient Estimation

It is clear that the proposed optimization scheme requires the estimation of the gradient of the failure probability functions. In what follows, a methodology for estimating the sensitivity of the failure probability functions in terms of the design variables is presented.

4.2.1 Performance Function Approximation

Recall that the failure domain Ω_{F_i} for a given design $\{x\}$ is defined as $\Omega_{F_i} = \{\{z\} \mid \kappa_i(\{x\}, \{z\}) \leq 0\}$, where $\{z\}$ is the vector of random variables that describes all

uncertainties involved in the problem. If $\{x_k\}$ is the current design, the performance function κ_i is approximated in the vicinity of the current design as

$$\bar{\kappa}_i(\{x\}, \{z\}) = \kappa_i(\{x_k\}, \{z\}) + \{\delta_i\}^T \{\Delta x\} \quad (28)$$

where $\{x\} = \{x_k\} + \{\Delta x\}$. For samples $\{z_j\}$, $j = 1, \dots, M$ near the limit state surface, that is, $\kappa_i(\{x\}, \{z_j\}) \approx 0$ the performance function is evaluated at points in the neighborhood of $\{x_k\}$. These points are generated as

$$\{x_{lk}\} - \{x_k\} = \{\Delta x\} = \frac{\{\xi_l\}}{\|\{\xi_l\}\|} R, \quad l = 1, \dots, N = Q \times M \quad (29)$$

where the components of the vector $\{\xi_l\}$ are independent, identically distributed standard Gaussian random variables, N and Q positive integers and R is a user-defined small positive number. This number defines the radius of the hypersphere $(\{\xi_l\} / \|\{\xi_l\}\|)R$ centered at the current design $\{x_k\}$. The coefficients $\{\delta_i\}$ of the approximation (28) are computed by least squares. To this end, the following set of equations is generated

$$\begin{aligned} \kappa_i(\{x_{lk}\}, \{z_j\}) &= \kappa_i(\{x_k\}, \{z_j\}) + \{\delta_i\}^T \frac{\{\xi_l\}}{\|\{\xi_l\}\|} R \\ l &= j + (q - 1) \times M, q = 1, \dots, Q, j = 1, \dots, M \end{aligned} \quad (30)$$

Since the samples $\{z_j\}$, $j = 1, \dots, M$ are chosen near the limit state surface the approximate performance function $\bar{\kappa}_i$ is representative of the behavior of the probability of the failure domain Ω_{F_i} in the vicinity of the current design $\{x_k\}$.

4.2.2 Sensitivity of Failure Probability Function

The gradient of the i -th failure probability function at $\{x_k\}$ can be estimated by means of the limit:

$$\left. \frac{\partial P_{F_i}(\{x\})}{\partial x_l} \right|_{\{x\}=\{x_k\}} = \lim_{\Delta x_l \rightarrow 0} \frac{P_{F_i}(\{x_k\} + \{u(l)\}\Delta x_l) - P_{F_i}(\{x_k\})}{\Delta x_l}, \quad l = 1, \dots, n_d \quad (31)$$

where $\{u(l)\}$ is a vector of length n_d with all entries equal to zero, except by the l -th entry, which is equal to one. Considering the definition of failure probability in Eq. 8, the above equation can be rewritten as:

$$\left. \frac{\partial P_{F_i}(\{x\})}{\partial x_l} \right|_{\{x\}=\{x_k\}} = \lim_{\Delta x_l \rightarrow 0} \left(\frac{P[\kappa_i(\{x_k\} + \{u(l)\}\Delta x_l, \{z\}) \leq 0]}{\Delta x_l} - \frac{P[\kappa_i(\{x_k\}, \{z\}) \leq 0]}{\Delta x_l} \right), \quad l = 1, \dots, n_d \quad (32)$$

where $P[\cdot]$ denotes probability of occurrence. Introducing Eqs. (7) and (28), this limit can be approximated as:

$$\left. \frac{\partial P_{F_i}(\{x\})}{\partial x_l} \right|_{\{x\}=\{x_k\}} \approx \lim_{\Delta x_l \rightarrow 0} \left(\frac{P[D_i(\{x_k\}, \{z\}) \geq 1 + \delta_{li} \Delta x_l]}{\Delta x_l} - \frac{P[D_i(\{x_k\}, \{z\}) \geq 1]}{\Delta x_l} \right), \quad l = 1, \dots, n_d \quad (33)$$

Thus, the calculation of the derivatives reduces to estimating the probability that the normalized demand (for a fixed design) exceeds two different threshold levels. These probabilities can be readily obtained from a single reliability analysis performed using, e.g. Subset Simulation. If the failure probability is approximated as an explicit function of the normalized demand in the vicinity of the limit state surface ($\kappa_i(\{x_k\}, \{z\}) = 0$) as

$$\begin{aligned} P[D_i(\{x_k\}, \{z\}) \geq D_i^*] &= P_{F_i}(D_i^*) \\ P_{F_i}(D_i^*) &\approx e^{\psi_0 + \psi_1(D_i^* - 1)}, \\ D_i^* &\in [1 - \epsilon, 1 + \epsilon] \end{aligned} \quad (34)$$

where D_i^* is a threshold of the normalized demand and ϵ represents a small tolerance, e.g. $\epsilon = 0.05$, then Eq. 33 can be simplified to

$$\left. \frac{\partial P_{F_i}(\{x\})}{\partial x_l} \right|_{\{x\}=\{x_k\}} \approx \lim_{\Delta x_l \rightarrow 0} \frac{e^{\psi_0 + \psi_1 \delta_{li} \Delta x_l} - e^{\psi_0}}{\Delta x_l} = \psi_1 \delta_{li} P_{F_i}(\{x_k\}), \quad l = 1, \dots, n_d \quad (35)$$

where δ_{li} is the l -th element of the vector $\{\delta_i\}$, and all other terms have been previously defined. Numerical experience has shown that the approximation introduced in Eq. 35 is adequate in the context of the proposed optimization scheme. Issues such as the number of points required for performing least square (Q and M), and the generation of design points in the vicinity of the current design (calibration of the radius R) are discussed in detail in reference [36].

4.2.3 Directional Sensitivity of Failure Probability Function

The approach proposed above can also be used for estimating the derivative along a given direction $\{y\}$, i.e.:

$$\nabla P_{F_i}(\{x\})^T \{y\} \Big|_{\{x\}=\{x_k\}} = \lim_{\Delta x \rightarrow 0} \frac{P_{F_i}(\{x_k\} + \{y\} \Delta x) - P_{F_i}(\{x_k\})}{\Delta x} \quad (36)$$

In this case, it suffices to generate an approximation of the performance function $\bar{\kappa}_i$ in the single-dimensional space defined by $\{y\}$. Thus, only one coefficient δ_i must

be computed for constructing the required approximation. Then, the least squares problem to be solved is (cf. Eq. 30)

$$\kappa_i(\{x_k\} + \{y\}\xi_j R, \{z_j\}) = \kappa_i(\{x_k\}, \{z_j\}) + \delta_i \xi_j R, \quad j = 1, \dots, N_d \quad (37)$$

where ξ_j , $j = 1, \dots, N_d$ are independent, identically distributed standard Gaussian random variables, $\{z_j\}$, $j = 1, \dots, N_d$ are samples of the uncertain parameters located near the limit state surface (i.e., $\kappa_i(\{x\}, \{z_j\}) \approx 0$), N_d is a positive integer and R is a user-defined small positive number. The number of evaluations of the performance function required for solving the least squares problem in Eq. 37 is smaller than the number of evaluations required to solve the problem in Eq. 30. Usually, the ratio between the number of samples required for solving the former and latter problems is $1/n_d$ (inversely proportional to the number of design variables). The directional derivative can be estimated as

$$\nabla P_{F_i}(\{x\})^T \{y\} \Big|_{\{x\}=\{x_k\}} \approx \psi_1 \delta_i P_{F_i}(\{x_k\}) \quad (38)$$

Finally, note that the approach for estimating the gradient and the directional derivative of the failure probability functions described in this section does not require any additional reliability analyses. Only the evaluation of the corresponding performance functions κ_i , $i = 1, \dots, n$ in the vicinity of the current design is needed.

4.3 Line Search

The step size problem, that is, the determination of the step to the nearest intersecting boundary (Sect. 3.4) is carried out as follows. If $\{x_k\}$ is the current design the idea of line search is to explore the design space along a single dimension. The one dimensional space based on $\{x_k\}$ is defined as

$$\Omega_{\{x_k\}} = \{\{x\} \mid \{x\} = \{x_k\} + \alpha\{d\}\} \quad (39)$$

4.3.1 Deterministic Constraints

For the deterministic linear constraints, the step to nearest intersecting boundary $\bar{\alpha}_s$ can be readily determined from the requirements that

$$-\epsilon \leq s_i(\{x_k\} + \alpha\{d\}) \leq 0, \quad i \notin J, \quad (40)$$

where ϵ is a positive user-defined scalar. For the deterministic nonlinear constraints, a step limit α_{limit} based on the lower and upper limits of the design variables is first determined from

$$x_i^l \leq x_{ik} + \alpha d_i \leq x_i^u, \quad i = 1, \dots, n_d \quad (41)$$

Next, all constraints $g_i, i = 1, \dots, m$ are evaluated at the point $\{x_k\} + \alpha_{limit}\{d\}$. If $g_{max} = \max_{i=1, \dots, m} \{g_i\} \leq 0$, then the step to the nearest deterministic nonlinear intersecting boundary is equal to $\bar{\alpha}_g = \alpha_{limit}$. Otherwise, a step α such that $0 < \alpha < \alpha_{limit}$ needs to be found. This step can be obtained by a bisection scheme [11].

4.3.2 Reliability Constraints

A bisection scheme could also be used for finding the step to the nearest intersecting boundary for the reliability constraints. However, this approach may imply several reliability analyses during line search. In order to reduce the number of failure probability estimations, an alternative approach is proposed in the present formulation. The approach consists in generating an approximate representation of the failure probabilities along the search direction using information on failure probabilities and – additionally – on directional derivatives at selected points. It should be noted that directional derivatives are used in the proposed approach because they can be calculated at small numerical costs once the failure probability has been computed at a given point. Moreover, in [21, 37], it has been shown that such information can substantially improve the quality of an approximate representation. The approximation of the failure probabilities along the search direction has the form:

$$\log(\bar{P}_{F_i}(\{x_k\} + \alpha\{d\})) = a_{k0} + a_{k1}\alpha + a_{k2}\alpha^2, \quad \alpha > 0 \quad (42)$$

where $\log(\cdot)$ indicates natural logarithm. This type of approximation has shown to be appropriate for solving RBO problems [10, 12]. The coefficients $a_{kl}, l = 0, 1, 2$ are calculated in two stages, i.e.:

1. Firstly, the failure probability and its derivative (along $\{d\}$) are calculated at the current candidate optimum design $\{x_k\}$ and also at a point $\{x^{initial}\} = \{x_k\} + \alpha_{initial}^{ki}\{d\}$; $\alpha_{initial}^{ki}, i = 1, \dots, n$ is defined as a percentage of the current design norm $\|\{x_k\}\|$, that is, $\alpha_{initial}^{ki} = \Gamma^{ki} \|\{x_k\}\|$. In this implementation, $\Gamma^{ki} \in [0.10, 0.25]$. With the information on probabilities and directional derivatives, the coefficients $a_{kl}, l = 0, 1, 2$ are determined using the approach proposed in [37] (see Appendix).
2. With the approximation generated at the previous stage, a new point $\{x^*\} = \{x_k\} + \alpha_*^{ki}\{d\}$ is located, such that $\bar{P}_{F_i}(\{x^*\}) = P_{F_i}^*$. Then, the failure probability and the directional derivative at $\{x^*\}$ are calculated. Finally, a new set

of coefficients a_{kl} , $l = 0, 1, 2$ is determined, using the information collected at $\{x_k\}$, $\{x^{initial}\}$ and $\{x^*\}$ (see Appendix).

Thus, using the approximation in Eq. 42, it is possible to find the step α_{h_i} for which the reliability constraint $h_i(\cdot)$ is active, i.e.:

$$P_{F_i}^* - \epsilon \leq \bar{P}_{F_i}(\{x_k\} + \alpha_{h_i}\{d\}) \leq P_{F_i}^* \quad (43)$$

where ϵ is a scalar that allows reliability constraints near the limit to be considered as active. Provided that the interpolation gives a good approximation of P_{F_i} along the search direction, the point $\{x_k\} + \alpha_{h_i}\{d\}$ is also an active design for the reliability constraint h_i . Then, the step to the nearest intersecting reliability constraint is given by $\bar{\alpha}_h = \text{Min}_{i=1, \dots, n} \alpha_{h_i}$. With the previous information, the nearest intersecting boundary from the current design $\{x_k\}$ is obtained from $\text{Min}\{\bar{\alpha}_s, \bar{\alpha}_g, \bar{\alpha}_h\}$. Validation calculations have shown that the number of function evaluations, including reliability analyses, is generally very small during the different cycles of the proposed optimization scheme.

4.4 Variability of Reliability Estimates

The reliability estimates carried out during the optimization process have associated certain variability. This variability is measured by the estimator coefficient of variation, i.e., the ratio of the standard deviation to the mean of the failure probability. In principle, the variability of the estimates can be controlled by the number of samples used during simulation. However, this alternative can be computationally involved especially when dealing with realistic problems. The average over a number of simulation runs can also be used to smooth the estimates variability during the optimization process. This is particularly important during the last iterations of the optimization scheme, i.e., near convergence. Numerical validations have shown that during the first cycles of optimization the average over different simulation runs is not essential since the algorithm global performance is not affected by the variability of the estimates.

The strategy proposed in Sect. 4.3 for approximating reliability constraints at the line search step has shown to be appropriate for coping with the variability of the reliability estimates in the numerical examples investigated in this contribution. As information on failure probabilities and directional derivatives (collected at three points) is used simultaneously, the resulting approximation is of sufficient accuracy.

Finally, as in the case of the reliability estimates, an accurate descent feasible direction is not strictly required during the first iterations. Only an approximate direction is sufficient since the important issue at this stage of the optimization process is approaching the vicinity of the optimum.

5 Application Problem

A ten-storey reinforced concrete (RC) shear frame including a base-isolation system (composed of non linear hysteretic devices) and subject to a stochastic ground acceleration is considered for study. The frame structure corresponds to one of the resistant element (axis B) of the structural model shown in Fig. 3. The details of the first three floors of the resistant element are also shown in the figure. Each floor of the RC frame is supported by six columns of square shape and a height of 3 m. It is assumed that the beams of the frame are rigid in the axial direction, so each floor can be described by a single horizontal degree of freedom (DOF); moreover, the base of the system is modeled as a rigid slab. Thus, the model involves a total of 11 DOFs. Moreover, it is assumed that the RC frame structure remains linear throughout the duration of the ground acceleration. The Young's modulus is equal to 3×10^{10} Pa. The tributary floor mass corresponding to axis B is equal to 1.5×10^5 kg, and the corresponding tributary mass of the base to axis B is equal to 3.0×10^5 kg. A 5% of

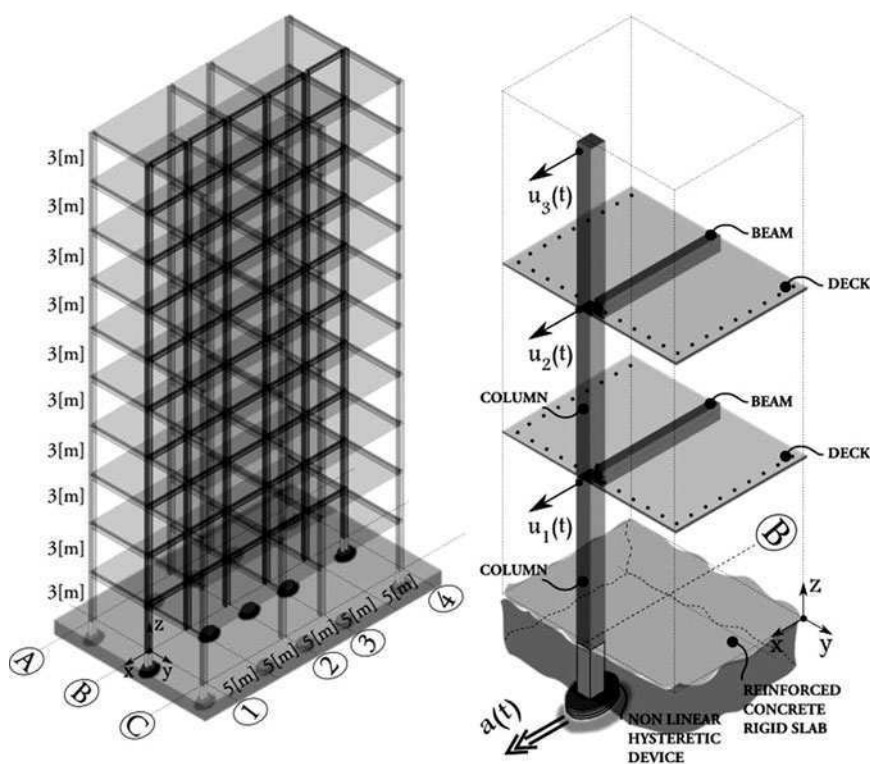


Fig. 3 Ten-storey RC shear frame with base-isolation system

critical damping is assumed in the model. The base-isolation system is composed of five non linear devices (NLD). Each of these devices follows the restoring force law

$$r(t) = k_d \left(u(t) - \gamma^1(t) + \gamma^2(t) \right), \quad (44)$$

where k_d denotes the initial stiffness of the device, $u(t)$ is the relative displacement between the base and the ground, and $\gamma^1(t)$ and $\gamma^2(t)$ denote the plastic elongations of the friction device. Using the supplementary variable $\tau(t) = u(t) - \gamma^1(t) + \gamma^2(t)$, the plastic elongations are specified by the nonlinear differential equations [28]

$$\begin{aligned} \dot{\gamma}^1(t) &= \dot{u}(t) H(\dot{u}(t)) \left[H(\tau(t) - \tau_y) \frac{\tau(t) - \tau_y}{\tau_p - \tau_y} H(\tau_p - \tau(t)) + H(\tau(t) - \tau_p) \right], \\ \dot{\gamma}^2(t) &= -\dot{u}(t) H(-\dot{u}(t)) \left[H(-\tau(t) - \tau_y) \frac{-\tau(t) - \tau_y}{\tau_p - \tau_y} H(\tau_p + \tau(t)) + H(-\tau(t) - \tau_p) \right], \end{aligned} \quad (45)$$

where $H(\cdot)$ denotes the Heaviside step function, τ_y is a parameter specifying the onset of yielding, and $k_d \tau_p$ is the maximum restoring force of the friction device. The values $\tau_p = 0.006$ m, $\tau_y = 0.0042$ m, and $k_d = 2.0 \times 10^8$ N/m are used in this case. The earthquake induced ground acceleration in the x -direction is modeled as a non-stationary filtered white noise process. The filter is defined by the first-order differential equation

$$\frac{d}{dt} \begin{Bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \\ w_4(t) \end{Bmatrix} = \begin{Bmatrix} 0 & 1 & 0 & 0 \\ -\Omega_1^2 & -2\xi_1\Omega_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \Omega_1^2 & 2\xi_1\Omega_1 & -\Omega_2^2 & -2\xi_2\Omega_2 \end{Bmatrix} \begin{Bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \\ w_4(t) \end{Bmatrix} + \begin{Bmatrix} 0 \\ \omega(t)e(t) \\ 0 \\ 0 \end{Bmatrix} \quad (46)$$

and the ground acceleration is defined as

$$a(t) = \Omega_1^2 w_1(t) + 2\xi_1\Omega_1 w_2(t) - \Omega_2^2 w_3(t) - 2\xi_2\Omega_2 w_4(t), \quad (47)$$

where $\omega(t)$ denotes white noise, and $e(t)$ is the envelope function defined as

$$e(t) = \begin{cases} t^2/16 & \text{if } t \in [0, 4 \text{ s}] \\ 1 & \text{if } t \in [4 \text{ s}, 10 \text{ s}] \\ e^{(t-10)^2} & \text{if } t \in [10 \text{ s}, 15 \text{ s}] \end{cases} \quad (48)$$

The values $\Omega_1 = 15.0$ rad/s, $\xi_1 = 0.6$, $\Omega_2 = 1.0$ rad/s, and $\xi_2 = 0.9$, and white noise intensity $I = 6.3 \times 10^{-2}$ m²/s³ have been used in this example. The sampling interval and the duration of the excitation are taken as $\Delta t = 0.01$ s, and $T = 15$ s, respectively. Then, the discrete-time white noise sequence $\omega(t_j) = \sqrt{I/\Delta t} z_j$, where $\{z_j\}$, z_j , $j = 1, \dots, 1,501$, are independent, identically distributed standard

Gaussian random variables is considered in this case. It is noted that the uncertainties involved in this problem are due only to the characterization of the stochastic excitation. That is, the structural parameters are assumed to be deterministic in this particular application.

The objective function C is defined as the area of the columns of the shear frame. The design variables $\{x\}$ are chosen as the inertia of the columns throughout the height, grouped in ten design variables, i.e. the inertia of the columns of each floor constitutes each of the design groups. The failure event is formulated as a first passage problem during the duration of the ground acceleration. The structural responses to be controlled are the ten interstorey drift displacements and the roof displacement (w.r.t. the base-isolation system). The threshold values are chosen equal to 0.2% of the floor height for the interstorey drift displacements and 0.1% of the frame height for the roof displacement. The target probability of failure (P_F^*) is set equal to 10^{-3} . Additionally, geometric and side constraints are incorporated in the problem. Thus, the reliability-based optimization problem is defined as:

$$\text{Min } C(\{x\})$$

subject to

$$\begin{aligned} P_F(\{x\}) - P_F^* &\leq 0, \\ x_{i+1} - x_i &\leq 0, \quad i = 1, \dots, 9 \\ -x_i + 2.1 \times 10^{-3} &\leq 0, \quad i = 1, \dots, 10 \\ x_i - 8.3 \times 10^{-2} &\leq 0, \quad i = 1, \dots, 10 \end{aligned} \quad (49)$$

The initial design is shown in Table 1. The results of the optimization procedure are presented in Fig. 4 in terms of the evolution of the objective function. Only a few optimization cycles are required for obtaining convergence. In fact, most of the improvement of the objective function takes place in the first three optimization

Table 1 Initial and final designs

Design variables	Initial design	Final design
x_1 (m ⁴)	5.47	4.75
x_2 (m ⁴)	5.47	4.75
x_3 (m ⁴)	5.47	4.75
x_4 (m ⁴)	5.47	4.69
x_5 (m ⁴)	5.47	4.56
x_6 (m ⁴)	3.41	3.79
x_7 (m ⁴)	3.41	3.57
x_8 (m ⁴)	3.41	3.01
x_9 (m ⁴)	3.41	2.15
x_{10} (m ⁴)	3.41	1.50
Probability of failure	$9.41 \cdot 10^{-7}$	$1.16 \cdot 10^{-3}$

Target failure probability: $1.00 \cdot 10^{-3}$

No. of optimization cycles: 8

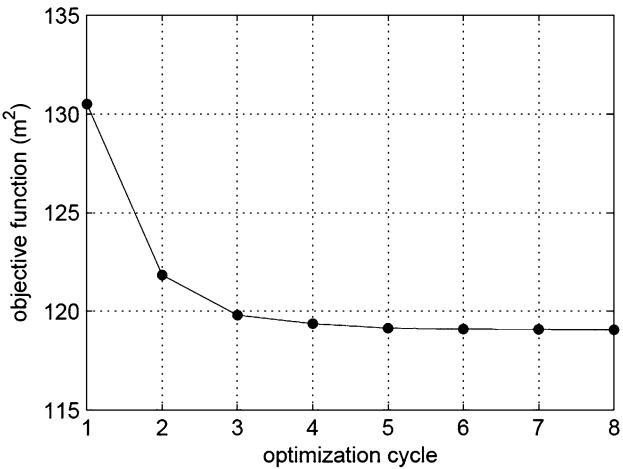


Fig. 4 Iteration history of the design process in terms of the objective function

Table 2 Summary of numerical efforts		
Type of analysis	Number of analyses required for solving RBO problem	Average number of simulations required per single analysis
Reliability	22	3,364
Gradient of failure probability	8	1,000
Directional derivative of failure probability	14	100

cycles. By construction the method generates a series of steadily improved feasible designs that moves toward the final design. This property is important from a practical viewpoint since the design process may be stopped at any stage still leading to acceptable feasible design better than the initial feasible estimate. The details on the optimization procedure for the initial design and the 8-th iteration cycle are summarized in Table 1.

The numerical efforts involved in the solution of the RBO problem are due to the estimation of the reliability and the estimation of its sensitivity, respectively. Table 2 summarizes the numerical costs associated with the solution of the RBO problem formulated above. The first column of this table indicates the type of analysis performed, while the second column shows the number of times the aforementioned analysis was repeated throughout the optimization procedure; finally, the third column indicates the average number of simulations required for performing one particular type of analysis. For example, a total of eight estimates of the gradient of the probability are required for solving the RBO problem; each of these analyses require 1,000 simulations for calibrating the approximate model shown in Eq. 28. Finally, the failure probability associated with the final design (see Table 1) is computed for the case where no base-isolation system is considered. The resulting

failure probability is $P_F = 4.50 \times 10^{-1}$. This result highlights the beneficial effect of the base-isolation system in protecting the superstructure (in this case, the ten-storey RC shear frame).

6 Conclusions

An approach for solving challenging reliability-based optimization problems involving dynamical systems under stochastic excitation has been presented. Numerical results have shown that the approach generates a sequence of steadily improved feasible designs. That is, the design process has monotonic convergence properties. This property is important from a practical view point since the optimization process can be stopped at any stage still leading to better designs than the initial feasible estimate. This is particularly attractive for dealing with involved problems such as reliability based optimization of dynamical systems under stochastic excitation. In these problems, which are the cases of interest in this work, each iteration of the optimization process is associated to high computational costs. Numerical experience has also shown that the algorithm converges in a relatively small number of optimization cycles. This in turn implies that only a moderate number of reliability estimates has to be performed during the entire design process. A critical issue for computing descent-feasible directions is shown to be the determination of the gradient of the failure probability functions. The approximation used in this contribution for estimating the gradient proved to be quite efficient. Concerning the line search step, the proposed approach for handling reliability constraints is considered to be most adequate, as the total number of reliability analyses per line search is limited to three. In addition, numerical results indicate that the approach allows to cope with the inherent variability of the reliability estimates as well.

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Appendix

In [37], a framework for constructing response surfaces (using information on both function values and derivatives) was proposed. From this framework, a particular formulation is applied for determining the coefficients of the approximation of the failure probabilities introduced in Eq. 42. Note that the formulation reported herein refers to a case involving single-dimensional independent variable; for the multidimensional case, it is referred to [37].

Assume that the value and derivative of the *exact* function $q(x)$ have been calculated at n_p points x_i , $i = 1, \dots, n_p$. The objective is determining the coefficients a_l , $l = 0, 1, 2$ of the polynomial:

$$\bar{q}(x) = a_0 + a_1x + a_2x^2 \quad (50)$$

The sought coefficients are determined by minimizing the following residual:

$$H(\{a\}) = H_0(\{a\}) + H_1(\{a\}) \quad (51)$$

The residuals $H_0(\cdot)$ and $H_1(\cdot)$ are defined as:

$$H_0(\{a\}) = \frac{\sum_{i=1}^{n_p} (q(x_i) - (a_0 + a_1x_i + a_2x_i^2))^2}{(\max_{i=1, \dots, n_p} (q(x_i)) - \min_{i=1, \dots, n_p} (q(x_i)))^2} \quad (52)$$

$$H_1(\{a\}) = \frac{4 \sum_{i=1}^{n_p} (q'(x_i) - (a_1 + 2a_2x_i))^2}{(|\max_{i=1, \dots, n_p} (q'(x_i))| + |\min_{i=1, \dots, n_p} (q'(x_i))|)^2} \quad (53)$$

where $q'(\cdot)$ denotes the first derivative of $q(\cdot)$.

References

1. Alexandrov, N.M., Dennis, J.E., Jr., Lewis, R.M., Torczon, V.: A trust-region framework for managing the use of approximation models in optimization. *Struct. Multidiscip. Optim.* **15**(1), 16–23 (1998)
2. Au, S.K.: Reliability-based design sensitivity by efficient simulation. *Comput. Struct.* **83**(14), 1048–1061 (2005)
3. Au, S.K., Beck, J.L.: Estimation of small failure probabilities in high dimensions by subset simulation. *Probabilist. Eng. Mech.* **16**(4), 263–277 (2001)
4. Bathe, K.J.: *Finite Element Procedures*. Prentice Hall, New Jersey (1996)
5. Beck, J.L., Au, S.K.: Reliability of dynamic systems using stochastic simulation. In: Soize, C., Schuëller, G.I. (eds.), *Structural Dynamics EUROLYN 2005 – Proceedings of the 6th International Conference on Structural Dynamics*, Vol. 1, pp. 23–30, Paris, France. Millpress, Rotterdam (2005)
6. Belegundu, A.D., Chandrupatla, T.R.: *Optimization Concepts and Applications in Engineering*. Prentice Hall, Upper Saddle River (1999)
7. Ching, J., Hsieh, Y.H.: Approximate reliability-based optimization using a three-step approach based on subset simulation. *J. Eng. Mech.* **133**(4), 481–493 (2007)
8. Ching, J., Hsieh, Y.H.: Local estimation of failure probability function and its confidence interval with maximum entropy principle. *Probabilist. Eng. Mech.* **22**(1), 39–49 (2007)
9. Enevoldsen, I., Sørensen, J.D.: Reliability-based optimization in structural engineering. *Struct. Saf.* **15**(3), 169–196 (1994)
10. Gasser, M., Schuëller, G.I.: Reliability-based optimization of structural systems. *Math. Method. Operat. Res.* **46**(3), 287–307 (1997)
11. Haftka, R.T., Gürdal, Z.: *Elements of Structural Optimization*. 3rd edn. Kluwer, Amsterdam (1992)
12. Jensen, H.A., Catalan, M.A.: On the effects of non-linear elements in the reliability-based optimal design of stochastic dynamical systems. *Int. J. Nonlinear Mech.* **42**(5), 802–816 (2007)

13. Jensen, H.A., Valdebenito, M.A., Schuëller, G.I.: An efficient reliability-based optimization scheme for uncertain linear systems subject to general Gaussian excitation. *Comput. Method. Appl. Mech. Eng.* **194**(1), 72–87 (2008)
14. Jensen, H.A., Ferré, M.S., Kusanovic, D.S.: Reliability-based synthesis of non-linear stochastic dynamical systems: a global approximation approach. *Int. J. Reliab. Saf.* **4**(2–3), 139–165 (2010)
15. Katafygiotis, L.S., Cheung, S.H.: Domain decomposition method for calculating the failure probability of linear dynamic systems subjected to Gaussian stochastic loads. *J. Eng. Mech.* **132**(5), 475–486 (2006)
16. Katafygiotis, L.S., Cheung, S.H.: Application of spherical subset simulation method and auxiliary domain method on a benchmark reliability study. *Struct. Saf.* **29**(3), 194–207 (2007)
17. Katafygiotis, L.S., Wan, J.: Reliability analysis of wind-excited structures using domain decomposition method and line sampling. In: Choi, C.-K. (ed.) 4th International Conference on Advances in Structural Engineering and Mechanics (ASEM 2008; CD-ROM), pp. 1024–1038, Jeju, Korea (2008)
18. Koutsourelakis, P.S.: Design of complex systems in the presence of large uncertainties: A statistical approach. *Comput. Method. Appl. Mech. Eng.* **197**(49–50), 4092–4103 (2008)
19. Koutsourelakis, P.S., Pradlwarter, H.J., Schuëller, G.I.: Reliability of structures in high dimensions, part I: Algorithms and applications. *Probabilist. Eng. Mech.* **19**(4), 409–417 (2004)
20. Kushner, H.J., Yin, G.G.: *Stochastic Approximation and Recursive Algorithms and Applications*. Springer, New York (2003)
21. Lauridsen, S., Vitali, R., van Keulen, F., Madsen, J.I., Haftka, R.T.: Response surface approximation using gradient information. In: *Proceedings 4th World Congress of Structural and Multidisciplinary Optimization*, Dalian, China (2001)
22. Luenberger, D.G.: *Introduction to Linear and Nonlinear Programming*. Addison-Wesley, Reading (1973)
23. Marti, K.: Limit load and shakedown analysis of plastic structures under stochastic uncertainty. *Comput. Method. Appl. Mech. Eng.* **198**(1), 42–51 (2008)
24. Melchers, R.E., Ahammed, M.: A fast approximate method for parameter sensitivity estimation in Monte Carlo structural reliability. *Comput. Struct.* **82**(1), 55–61 (2004)
25. Papadrakakis, M., Lagaros, N.D.: Reliability-based structural optimization using neural networks and Monte Carlo simulation. *Comput. Method. Appl. Mech. Eng.* **191**(32), 3491–3507 (2002)
26. Papadrakakis, M., Lagaros, N.D., Plevris, V.: Design optimization of steel structures considering uncertainties. *Eng. Struct.* **27**(9), 1408–1418 (2005)
27. Pellissetti, M.F., Schuëller, G.I., Pradlwarter, H.J., Calvi, A., Fransen, S., Klein, M.: Reliability analysis of spacecraft structures under static and dynamic loading. *Comput. Struct.* **84**(21), 1313–1325 (2006)
28. Pradlwarter, H.J.: Deterministic integration algorithms for stochastic response computations of FE-systems. *Comput. Struct.* **80**(18–19), 1489–1502 (2002)
29. Rosen, J.: The gradient projection method for nonlinear programming, I. linear constraints. *J. Soc. Indust. Appl. Math.* **8**, 181–217 (1960)
30. Royset, J.O., Polak, E.: Reliability-based optimal design using sample average approximations. *Probabilist. Eng. Mech.* **19**(4), 331–343 (2004)
31. Schuëller, G.I. (ed.): Computational methods in optimization considering uncertainties – Special Issue. *Comput. Method. Appl. Mech. Eng.* **198**(1), 1–164 (2008)
32. Schuëller, G.I., Jensen, H.A.: Computational methods in optimization considering uncertainties – An Overview. *Comput. Method. Appl. Mech. Eng.* **198**(1), 2–13 (2008)
33. Schuëller, G.I., Pradlwarter, H.J., Koutsourelakis, P.S.: A critical appraisal of reliability estimation procedures for high dimensions. *Probabilist. Eng. Mech.* **19**(4), 463–474 (2004)
34. Spall, J.C.: *Introduction to Stochastic Search and Optimization. Estimation, Simulation and Control*. Wiley, New York (2003)
35. Taflanidis, A.A., Beck, J.L.: Stochastic subset optimization for optimal reliability problems. *Probabilist. Eng. Mech.* **23**(2–3), 324–338 (2008)

36. Valdebenito, M.A., Schuëller, G.I.: Efficient strategies for reliability-based optimization involving non linear, dynamical structures. IfM-Int.Work.Report No.: 3-61, Institute of Engineering Mechanics, University of Innsbruck, Austria, EU (2009)
37. van Keulen, F., Vervenne, K.: Gradient-enhanced response surface building. *Struct. Multidiscip. Optimization*, **27**(5), 337–351 (2004)
38. Wu, Y.T.: Computational methods for efficient structural reliability and reliability sensitivity analysis. *AIAA J.* **32**(7), 1717–1723 (1994)
39. Zienkiewicz, O., Taylor, R.: *The Finite Element Method*, Volumes I, II and III, 5th edn. Butterworth-Heinemann, Oxford (2000)

Application of a Mode-based Meta-Model for the Reliability Assessment of Structures Subjected to Stochastic Ground Acceleration

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Abstract Realistic modeling of structures depends on both refined FE models and the incorporation of uncertainties inherent to geometric, structural and loading parameters. In this work, a mode-based meta-model is adopted for performing an efficient stochastic dynamic reliability analysis of a multi-story building subjected to stochastic ground acceleration. The meta-model allows for an efficient evaluation of the structural modes which are represented as a linear combination of the modes of a reference solution. The calibration of the meta-model is based on the modes of a few, say 50, samples of the structure obtained with Monte Carlo simulation. The parameters of the meta-model are then evaluated by the least square solution of the parameters determined for each sample of the calibration set. The reliability analysis is carried out efficiently through the application of the meta-model for estimating the modal properties which are then subsequently required to perform a mode-superposition analysis. The verification of the results for the reliability assessment carried out with the meta-model is performed with a full finite element analysis and shows a good agreement of the response over the whole duration of the stochastic ground acceleration.

1 Introduction

Approximate and at the same time mathematically simple models of the simulation model, so-called meta-models or surrogate models [5], play a central role in many scientific applications. The use of such surrogate models aims to considerably reduce the computational cost and is most suitable when a large number of simulations needs to be performed. In fact, this is especially the case for the response assessment of structural systems, e.g. considering reliability analysis, uncertainty quantification, etc.

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When a realistic treatment of a structural dynamics problem is of interest, uncertainty modeling of both the structural parameters as well as the loading process needs to be considered. Within this work the modeling of uncertainties is done in the framework of classical probability theory. This means that geometric and structural parameter uncertainties are modeled as random variables and/or random fields and the uncertain loading is modeled as a stochastic process. Classical probability theory is especially meaningful when enough data is available for describing the uncertainties statistically. This is appropriate for parameters such as the Young's modulus, wall or column thicknesses, dead loads, etc. Also, the earthquake loading can be represented realistically for certain areas of the world where enough measurement data have been recorded over the years.

However, in some cases, other concepts could be useful for treating uncertainties, as for example when dealing with model uncertainties, which cannot be simply handled in a similar fashion as parameter uncertainties. Thus, it has been proposed to deal with model uncertainties e.g. by using a recently introduced non-parametric approach [16, 17]; subjective knowledge or insufficiently known parameters might be treated by adopting fuzzy analysis [7, 8] and when measured data is available or when the system is steadily monitored, then Bayesian analysis [2, 3] is a viable analysis method. In general, all these differing approaches will provide values – e.g. for the structural safety – which need to be assessed critically. However, such a comparison is not a simple task, because it requires deep understanding and a correct interpretation of the various approaches.

In stochastic structural dynamics, each sample – i.e. for the structural parameters and the loading – can be treated independently in a deterministic manner when Monte Carlo simulation is applied. For large FE models, the deterministic dynamic solution is computationally already demanding. The solution can be obtained either with modal analysis or numerical integration. It depends on the specific problem which method is more convenient for application [1]. Both methods are applicable for linear and non-linear analysis as well. In case of large finite element structures modal analysis is computationally more efficient. When dealing with small or local non-linearities, it is possible to reformulate the equation of motion such that modal analysis is still applicable. The idea is to split the damping and restoring forces in a linear and non-linear part, whereas the non-linear forces are moved to the right hand side of the equation and then solved in an iterative manner (see e.g. [12–14]).

However, in case of non-linear analysis, numerical integration procedures outperform modal analysis regarding computational efficiency. This is due to the fact that in case of modal analysis in non-linear dynamic analysis the mode shapes need to be continuously updated. The updating procedure might be done most suitably with the subspace iteration methodology [1]. However, this approach is only applicable, when the modes need not to be updated too frequently.

In this paper, the first method, i.e. modal analysis is considered for the dynamic analysis of linear structures. Thus, for every simulation, the associated generalized eigenvalue problem has to be solved, which is computationally demanding for large FE models. In order to reduce the computational costs a meta-model for the modes of the structure is adopted. In this work, a slight variation when compared with the

meta-model for the eigensolution which has been introduced in [10], is applied. So far, the meta-model has been applied for investigations on the FRF-variability (see e.g. [9, 15]). FRFs are nonlinear functions of the modes which are approximated with a linear combination of the modes of a reference solution. In this paper, applications to the reliability assessment based on the meta-model for the modes are reported. For this purpose (1) the top-floor displacement due to a typical sample of a stochastic ground acceleration in the horizontal direction and (2) the relative displacement between story 1 and 2 in case of stochastic ground acceleration are analyzed. Linear structural dynamics is considered throughout this study. Serviceability and not structural degradation or collapse is assumed as criterion for the reliability assessment.

2 Meta-Model for the Structural Modes

2.1 General Aspects

The variability of the mode shapes due to parameter uncertainties can be represented as a function of neighboring modes of a reference system. In many cases the nominal solution, i.e. the solution for the mean value of the vector of uncertain parameters, might be used for this purpose. Alternatively, also multiple reference solutions could be used for the enrichment of the mathematical basis. It is assumed, that the k -th mode shape can be described by a linear combination of m mode shapes of the reference solution as shown in Fig. 1:

$$\phi_k^{(j)} = \sum_{i=k-m_k}^{k+m_k} \alpha_{ki}^{(j)} \phi_i^{(0)}, \quad (1)$$

where $\Phi^{(0)}$ denotes the modal matrix containing *all* modes of the reference system, $\phi_k^{(j)}$ the k -th mode of simulation j and α_k contains the $m = (2m_k + 1)$ coefficients

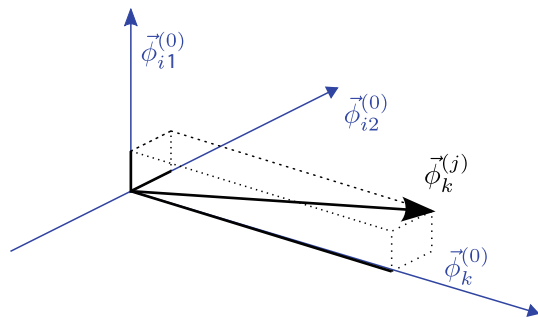


Fig. 1 Schematic sketch of the rotation in the space of the reference solution $\Phi^{(0)}$

of the linear combination. The associated $(2m_k + 1)$ modes of the nominal solution $\Phi^{(0)}$ are denoted as

$$\Phi_k^{(0)} = \left[\phi_{k-m_k}^{(0)}, \phi_{k-m_k+1}^{(0)}, \dots, \phi_k^{(0)}, \dots, \phi_{k+m_k-1}^{(0)}, \phi_{k+m_k}^{(0)} \right]. \quad (2)$$

The number $(2m_k + 1)$ of modes involved in the linear combination increases with frequency. Few, say two to four modes, generally suffice to describe the first group of modes. With increasing frequency, the density of eigenfrequencies increases and 10 or more modes should be employed for an accurate approximation.

Consequently, the vector α_k is determined by the solution of the least square problem of the associated linear relation in Eq. 1, which in its normal form is given by

$$\alpha_k = \left[\left(\Phi_k^{(0)} \right)^T \Phi_k^{(0)} \right]^{-1} \left(\Phi_k^{(0)} \right)^T \phi_k^{(j)}. \quad (3)$$

As shown in Fig. 1, the mode $\phi_k^{(j)}$ can be interpreted as a rotation of the mode $\phi_k^{(0)}$ in the modal space of the reference solution $\Phi_k^{(0)}$. Moreover, the length of vector $\phi_k^{(j)}$ might also slightly differ from the length of $\phi_k^{(0)}$.

The vector $\alpha_k = \{\alpha_{k,i}\}_{i=k-m_k}^{k+m_k}$ can be seen as the projection of $\phi_k^{(j)}$ onto the base vectors of $\Phi_k^{(0)}$. Then, the Euclidean norm $a_k = \|\alpha_k\|$ determines the ratio $\|\phi_k^{(j)}\| / \|\phi_k^{(0)}\|$. In the simplest case, where small rotations of $\phi_k^{(j)}$ can be assumed for small variations of the uncertain structural parameters, the factor $\alpha_{k,k}$ will be close to 1 and all factors $\alpha_{k,i \neq k}$ will be small and hence can be approximated sufficiently well by a linear function of the uncertain parameters.

The corresponding eigenfrequency is given by

$$\omega_k^{(j)} = \left(1 + \Delta\omega_k^{(j)} \right) \omega_k^{(0)}, \quad \Delta\omega_k^{(j)} = \left(\omega_k^{(j)} / \omega_k^{(0)} - 1 \right). \quad (4)$$

Each mode can then be represented by

$$\phi_k^{(j)} = a_k \sum_{i=k-m_k}^{k+m_k} \hat{\alpha}_{k,i}^{(j)} \phi_i^{(0)}. \quad (5)$$

The modal properties of each mode, i.e. $\omega_k^{(j)}$ and $\phi_k^{(j)}$ are then specified as a function of $a_k^{(j)}$, $\Delta\omega_k^{(j)}$, $\hat{\alpha}_k^{(j)}$ and the nominal solution $\omega_k^{(0)}$ and $\Phi_k^{(0)}$.

2.2 Meta-Model for the Modal Quantities

Let $\mathbf{Y}$ be a $N \times M$ matrix where N is the number of independent random samples $j = 1, 2, \dots, N$ and $M = \sum_{k=1}^K (2m_k + 3)$ is the number of factors describing the

modal properties of all considered K modes. Hence, the j -th row of $\mathbf{Y}$ contains the quantities

$$\mathbf{Y}^{(j)} = \left[a_1^{(j)}, \Delta\omega_1^{(j)}, \hat{\boldsymbol{\alpha}}_1^{(j)T}, \dots, a_K^{(j)}, \Delta\omega_K^{(j)}, \hat{\boldsymbol{\alpha}}_K^{(j)T} \right]. \quad (6)$$

Let further $\mathbf{X}$ be a $N \times n$ matrix which comprises in each row the vector $\mathbf{x}^{(j)}$ of realizations of the n uncertain parameters. The task of the meta-model can then be described as establishment of an approximate functional relation between the matrices $\mathbf{X}$ and $\mathbf{Y}$ respectively,

$$\mathbf{Y}(\mathbf{X}) = f(\mathbf{X}), \quad (7)$$

which allows for any vector $\mathbf{x}$ to establish the associated vector $\mathbf{y}(\mathbf{x})$:

$$\mathbf{y}(\mathbf{x}) = \left[a_1(\mathbf{x}), \Delta\omega_1(\mathbf{x}), \hat{\boldsymbol{\alpha}}_1^T(\mathbf{x}), \dots, a_K(\mathbf{x}), \Delta\omega_K(\mathbf{x}), \hat{\boldsymbol{\alpha}}_K^T(\mathbf{x}) \right]. \quad (8)$$

The simplest relation between $\mathbf{x}$ and $\mathbf{y} = \mathbf{y}(\mathbf{x})$ is a linear relation of the form

$$\mathbf{y} = \mathbf{S}\mathbf{x}, \quad (9)$$

in which $\mathbf{S}$ is a constant matrix of size $M \times n$. The matrix $\mathbf{S}$ needs to be established by exploring the available information given by the two matrices $\mathbf{X}$ and $\mathbf{Y}$.

Solution of the α_i is obtained with the following least square problem

$$\boldsymbol{\alpha}_i = \left(\mathbf{X}^T \mathbf{X} \right)^{(-1)} \mathbf{X}^T \mathbf{Y}. \quad (10)$$

The required sample size N to establish this linear relation grows only linearly with the number n of uncertain input parameters.

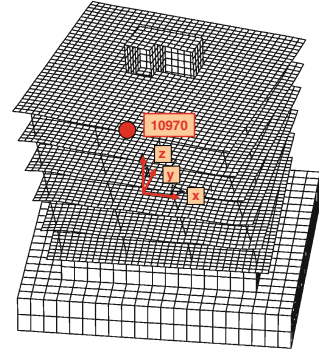
To be useful in linear dynamic analysis, the number of simulations required for calibration of the meta-model should be as small as possible. For many applications, a reduced set of 10–15 modes, i.e. ~ 40 –45 parameters for the meta-model, suffices in order to approximate accurately the required response and $N = 50$ simulations is a reasonable and affordable size for the calibration set.

3 Numerical Example

3.1 Model Description

A 6-story building (see Fig. 2), modeled with the finite element code FEAP [6], is considered in this numerical example. The finite element model consists of 9,543 elements and contains mainly shell and beam elements. The shell elements are used

Fig. 2 Finite element model of the 6-story building: first mode shape



for modeling all two-dimensional parts such as the walls and the floor slabs, the beam elements represent the columns and solid elements provide the discretization of the ground. In total, there are 53,364 degrees of freedom.

The interaction of the building with the ground is approximately modeled by using a 4.4 m thick layer of solid elements with a mass density of $1,800 \text{ kg/m}^3$ and a Young's modulus of $5 \cdot 10^9 \text{ N/m}^2$. The ground floor consists of circumferential walls, whereas the remaining five stories consist of 4×4 columns positioned on a quadratic grid of $7.5 \times 7.5 \text{ m}$. The main lateral and rotational stiffness of the building arises from the staircase. The Young's modulus of the concrete is assumed as $3.5 \cdot 10^{10} \text{ N/m}^2$, the mass density as $2,500 \text{ kg/m}^3$, the thicknesses of the walls and floor slabs measure 0.2 m. The plates representing the stairs are 0.16 m thick. The height of each story is 3.6 m, leading to a total height of 18 m for the building.

3.1.1 Uncertainty Modeling

The Young's moduli E_i of the various structural components of the building are modeled with Gaussian distributed random variables (RV). Three random variables describe the Young's modulus E of the stairs (RV_2), the floors (RV_3) and the columns (RV_4) over the six-stories. One RV is used to model the correlation between the vertical walls (RV_1) over the six-stories. In addition $3 \times 6 = 18$ RVs (RV_5 - RV_{22}) are used for the vertical wall elements of each story. A more detailed modeling of the vertical wall elements of the six-story building is necessary, because the variability of the horizontal displacement obviously depends on these parts of the structure. Thus, the Young's modulus for the walls is obtained as the summation of two random variables.

The uncertain Young's moduli are then given by:

$$E_i = \begin{cases} \text{RV}_i & i = 2, \dots, 4 \\ \text{RV}_1 + \text{RV}_i & i = 5, \dots, 22 \end{cases} \quad (11)$$

The coefficient of variation of the RVs is assumed to be 5%.

3.2 *Dynamic Analysis of the Nominal System*

3.2.1 **Modal Analysis**

The solution of the structural eigenvalue problem leads to the following eigen-frequencies:

Mode #	1	2	3	4	5	6	7	8	...	13	14
Frequency [Hz]	0.97	1.81	2.97	3.43	6.63	6.79	7.06	7.31	...	7.64	7.72

The deflection of the first mode is shown in Fig. 2.

3.2.2 **Modal Contribution**

The computational efforts can be considerably reduced when mode superposition analysis is applied. In general, a few modes, i.e. less than 10, suffice for accurately approximating the displacement response due to stochastic ground acceleration.

The displacement in x-direction at the top-floor level (node 10,970, see Fig. 2) represents the maximum displacement of the nominal structure due to the given sample of the stochastic loading. The ground acceleration used acts in the global x-direction of the building and is shown in Fig. 3. The contribution of the first mode to the total response constitutes approximately 95%. The contribution of the distinct modes is presented in Fig. 4. Thus, for the following computation five modes will be considered for the meta-model.

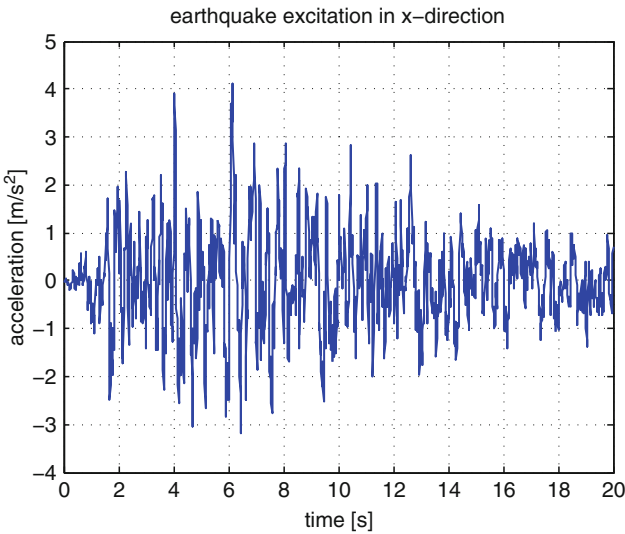


Fig. 3 Sample of stochastic ground acceleration

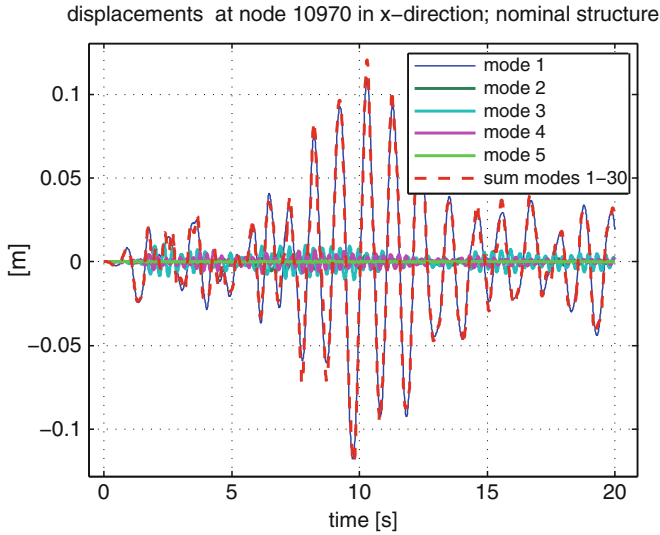


Fig. 4 Displacement at top floor due to stochastic ground acceleration

3.3 Set-up of the Meta-Model

The parameters of the meta-model have been evaluated by solving the least square problem (c.f. Eq. 10) based on 50 samples of the associated structural eigenvalue problem. The number of parameters used for the meta-model of the six-story building was chosen to be 24. The number of samples used for this example fully suffices for calibrating the meta-model and is verified by the good correspondence of the structural response obtained (1) with the full finite element solution and (2) with the approximate solution by the meta-model.

The structural matrices are assembled by using the finite element code FEAP. Then the eigenvalue problem is solved with MATLAB and the five modes are associated with modes $\Phi^{(0)}$ of the reference system.

3.4 Reliability Assessment Using a Mode-based Meta-Model

3.4.1 Case 1: Deterministic Loading

In this example, the structural reliability is assessed for one typical sample of the ground acceleration as shown in Fig. 3. The acceleration is acting only in the global x-direction of the building. The influence of the structural uncertainties is investigated by direct Monte Carlo simulation when using the mode-based meta-model.

The response quantity of interest is the maximum displacement in x-direction at the top-floor level which is reached for node 10,970. In this example, this response quantity is assumed to be the critical response quantity, jeopardizing the serviceability of the building.

Assessment of the Correctness of the Analysis Based on the Meta-Model

The correctness of the meta-model has been checked by comparing the solution obtained by using the meta-model and the FE-solution. Thus, 10^4 samples are generated with the meta-model and the associated displacement response is determined. The response among the calculated 10^4 samples with the maximum value of the displacement is denoted as $\mathbf{r}_{max}$. The associated vector with the values for the uncertain structural parameters is represented by $\mathbf{x}_{max}$. This means that the structural configuration with the parameters $\mathbf{x}_{max}$ leads to the highest observed displacement $\mathbf{r}_{max}$ within the 10^4 samples used (see Fig. 5). For all samples, the relative difference of the maximum response results in $\sim 2\text{--}3\%$. Figure 6 shows this comparison for the sample $\mathbf{x}_{max}$ and 2 random samples $\mathbf{x}_a$ and $\mathbf{x}_b$ for the time $[0, 12]$ s.

Reliability Assessment by Meta-Model

The reliability assessment of the top floor displacement (node 10,970, x-direction) has been carried out by performing MCS based on 10^4 samples. The quantiles and

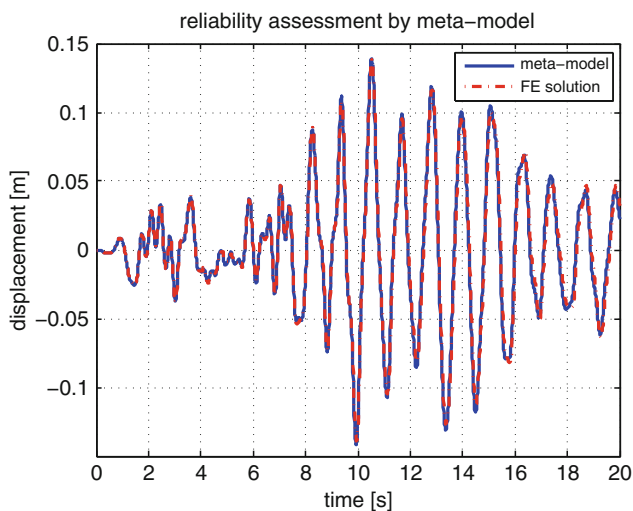


Fig. 5 Comparison of the maximum response

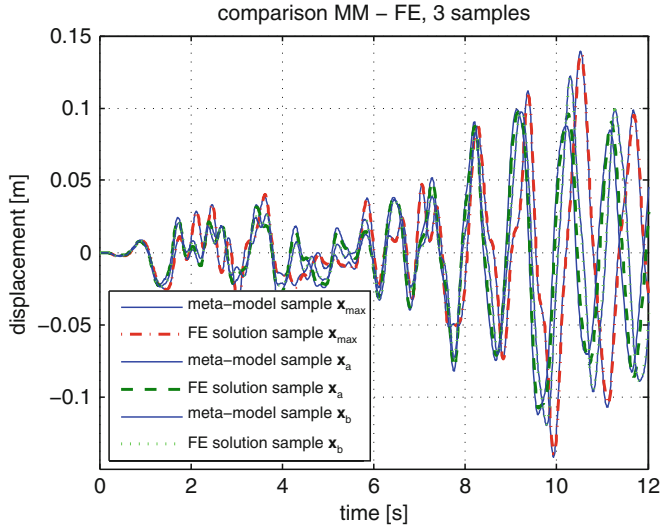


Fig. 6 Comparison of three samples over the time [0,12] s

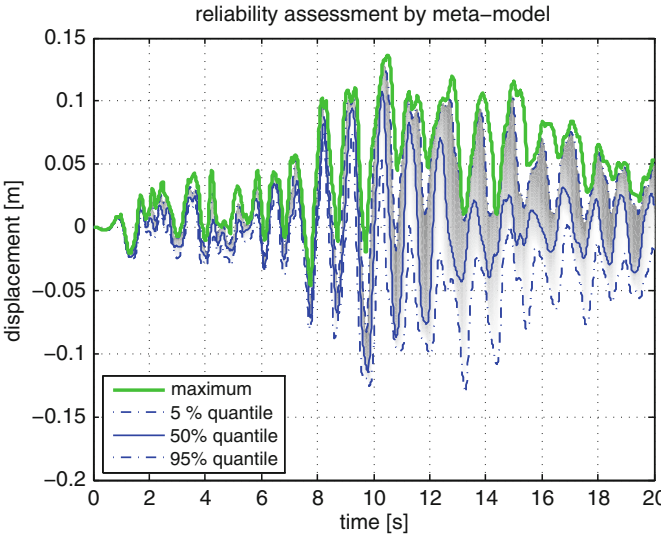


Fig. 7 Reliability assessment – quantiles of the response

the envelope of the maximum responses obtained are shown in Fig. 7. The failure probability for the displacement level has been evaluated and is shown together with its standard deviation σ in Fig. 8.

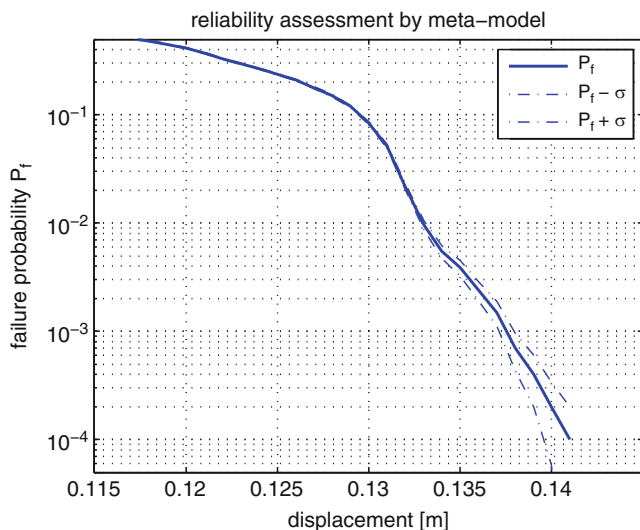


Fig. 8 Reliability assessment – failure probability P_f

3.4.2 Case 2: Stochastic Ground Acceleration

Generation of Samples of the Stochastic Ground Acceleration

For the assessment of the structural response due to stochastic ground acceleration, in many cases recorded earthquake records are employed. These records are then used for statistical assessment of the occurring earthquakes. Useful information such as the determination of the frequency content of the earthquakes, properties of the soil, propagation from the seismic center, etc. are gained. This information is further processed (e.g. scaling, etc.) and then by application of direct Monte Carlo simulation, the reliability of the structure due to inherent structural uncertainties is determined.

In Sect. 3.4.1 – Case 1 the meta-model is applied for the reliability assessment of the 6-story building due to a typical stochastic ground acceleration. However, when Monte Carlo simulation is performed, it is reasonable also to incorporate uncertainties on the loading side which highly affects the structural response. This can be done in a straightforward manner.

Digital realizations of the stochastic ground acceleration (simulated earthquake records) can be obtained by the so-called Karhunen–Loève expansion from the available earthquake data. After the evaluation of the covariance matrix of the accordingly scaled records, it is trivial to obtain the samples. This approach is not used within this paper.

A further option is to establish the covariance matrix such that it is compatible to some response spectra requirements. In this work, an approach based on filtered white noise is used. To cover the unknown frequency content and random

amplitudes of the acceleration, the model proposed in [4] is applied, where the acceleration is represented as the output of the response of a linear filter excited by white noise

$$a(t) = \Omega_g^2 v_1(t) + 2\xi_g \Omega_g v_2(t) - \Omega_f^2 v_3(t) - 2\xi_f \Omega_f v_4(t), \quad (12)$$

in which Ω_g represents the dominant frequency of the ground and Ω_f ensures that the spectrum of the acceleration tends to zero for frequencies approaching zero. The linear filter is described by the differential equation

$$\frac{d}{dt} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\Omega_g^2 & -2\xi_g \Omega_g & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \Omega_g^2 & 2\xi_g \Omega_g & -\Omega_f^2 & -2\xi_f \Omega_f \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} + \begin{bmatrix} 0 \\ w(t) \\ 0 \\ 0 \end{bmatrix} \quad (13)$$

and the modulating function $h(t)$ of the white noise intensity is given as

$$h(t) = I_0 \begin{cases} 0 & t = 0\text{s} \\ t/2 & 0\text{s} \leq t \leq 2\text{s} \\ 1 & 2\text{s} \leq t \leq 10\text{s} \\ \exp -0.1(t - 10) & t \geq 10\text{s} \end{cases}, \quad (14)$$

where $w(t)$ stands for a Gaussian white noise with auto-correlation function $E[w(t)w(t+s)] = I(t)/(2\pi)$. $I(t)$ denotes the intensity of the white noise which equals $I(t) = I_0$ in case of constant white noise excitation. In this example the modulating function $h(t)$ is multiplied with the white noise in order to represent the non-stationarity of the ground motion. The values $\Omega_g = 15.0$ rad/s, $\xi_g = 0.8$, $\Omega_f = 0.3$ rad/s, $\xi_f = 0.995$, and $I_0 = 0.18$ m²/s³ have been used to model the filter. To obtain a unique acceleration of the filter, independent normally distributed impulses are applied. The impulse at each discrete time step $t_k = k\Delta t$ has zero mean and standard deviation $\sqrt{I(k\Delta t)\Delta t}$.

When a large number of realizations is required, it is computationally more convenient to apply the approach based on the Karhunen–Loève expansion rather than to integrate the filter equation directly [11]. First, the impulse response function for the acceleration $a_{IRF}(t)$ (Eq. 12) is computed by using the linear filter (Eq. 13). For this purpose the linear filter is integrated for free motion with zero initial displacement, and $\dot{v}_1(0) = \dot{v}_3(0) = \dot{v}_4(0) = 0$ and $\dot{v}_2(0) = 1$.

To establish the covariance matrix, a single impulse at time $k\Delta t$ is considered. Then the covariance matrix of $a_{IRF}(t)$ is given by

$$C_{ij}(k) = \Delta t \cdot I(k\Delta t) \begin{cases} a_{IRF}((i-k)\Delta t) a_{IRF}((j-k)\Delta t) & \text{for } i \geq k \text{ and } j \geq k \\ 0 & \text{for } i < k \text{ or } j < k \end{cases} \quad (15)$$

and

$$C_{ij} = \sum_{k=0}^{2001} C_{ij}(k). \quad (16)$$

Finally, the eigenvectors and eigenvalues of the covariance matrix are evaluated and the Karhunen–Loève expansion is applied for determining realizations of the stochastic ground acceleration.

$$a_i(t) = \sum_{j=1}^{N_{KL}} \phi_j \sqrt{\lambda_j} \xi_i, \quad (17)$$

In this paper, 300 Karhunen–Loève terms are used.

Set-up of the Meta-Model

The meta-model which has been calibrated as described in Sect. 3.4.1 is also adopted for this second study. Since the structural model has not been changed, the meta-model does not change. The difference in this second example is only in the acceleration, which is now an independently sampled acceleration for any simulation and is acting in both the horizontal x and y directions. Each stochastic ground acceleration is a unique realization of a stochastic process according to Eq. 12.

Assessment of the Correctness of the Analysis Based on the Meta-Model

The used meta-model has been verified in Sect. 3.4.1 and is also valid for this case, because the response quantity is the relative displacement of the front left corner of story 1 and 2. The according comparison is visualized in Fig. 9.

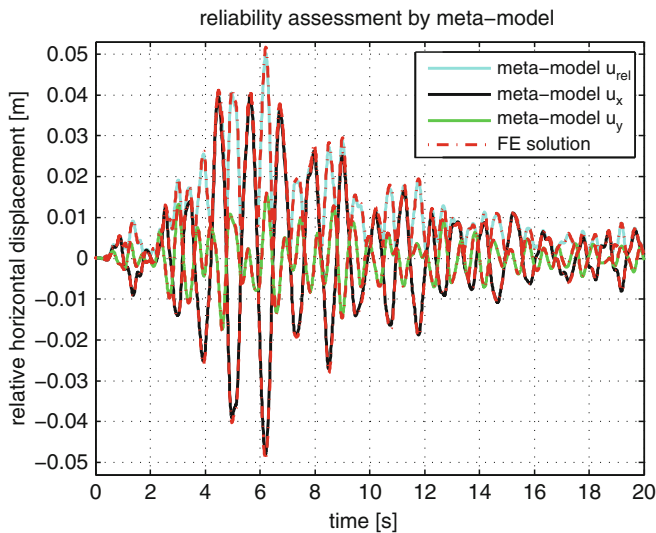


Fig. 9 Comparison of the response: Meta-model and finite element analysis

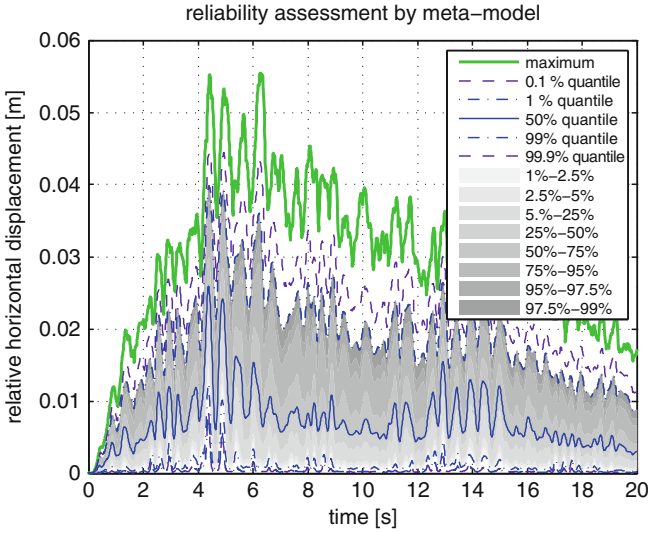


Fig. 10 Reliability assessment – quantiles of the response

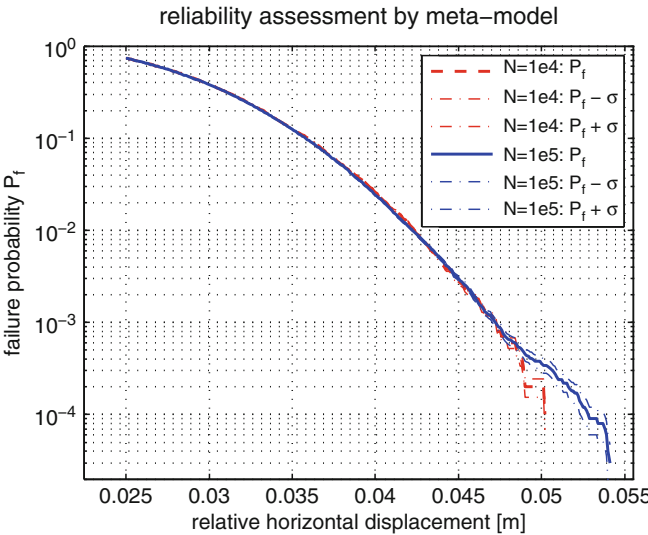


Fig. 11 Reliability assessment – failure probability P_f

Reliability Assessment by Meta-Model

The reliability assessment has been carried out by performing MCS based on 10^4 and 10^5 samples. The quantiles and the envelope of the maximum responses obtained are shown in Fig. 10.

The failure probability, i.e. the probability that the relative displacement exceeds a certain threshold level, has been evaluated and is illustrated in Fig. 11. The statistical variation of the failure probability is indicated by the standard deviation σ .

4 Conclusions

A mode-based meta-model has been used to perform an approximate, but very accurate reliability assessment of a six-story building. Only 50 samples were used for the calibration of the meta-model, whereas the failure probability has been computed based on 10^4 and 10^5 samples.

The main savings in computational time are due to the fact that the structural eigenvalue problem needs to be solved only for the 50 samples of the calibration set. The modes and eigenfrequencies of the samples created with the meta-model are given by the established linear relation.

In addition, application of the mode superposition analysis showed that the first few modes suffice in order to represent the displacement response accurately.

For future applications, the evaluation of maximum stresses in the structure are of interest. For such cases, few modes do not suffice anymore for an accurate response evaluation. This causes a larger computational cost, when calibrating the meta-model. Moreover, the mode-mixing phenomenon is expected to hamper the association of the modes in the calibration set to the reference mode set $\Phi^{(0)}$.

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References

1. Bathe, K.J.: Finite Element Procedures. Prentice Hall, Upper Saddle River (1996)
2. Beck, J.L., Au, S.-K.: Bayesian updating of structural models and reliability using Markov Chain Monte Carlo simulation. *J. Eng. Mech.* **128**(4), 380–391 (2002)
3. Beck, J.L., Katafygiotis, L.S.: Updating models and their uncertainties. I: Bayesian statistical framework. *J. Eng. Mech.* **124**(4), 455–461 (1998)
4. Clough, R.W., Penzien, J.: Dynamics of Structures, 2nd edn. McGraw-Hill, New York (1993)
5. Kleijnen, J.P.C., Sargent, R.G.: A methodology for fitting and validating metamodels in simulation. *Eur. J. Oper. Res.* **120**(1), 14–29 (2000)
6. FEAP: A finite element analysis program. University of California at Berkeley. <http://www.ce.berkeley.edu/feap/>
7. Moens, D., Vandepitte, D.: A survey of non-probabilistic uncertainty treatment in finite element analysis. *Comput. Method. Appl. Mech. Eng.* **194**(12–16), 1527–1555 (2005)
8. Möller, B., Beer, M.: Fuzzy Randomness – Uncertainty in Civil Engineering and Computational Mechanics. Springer, Berlin/Heidelberg/New York (2004)

9. Pichler, L., Pradlwarter, H.J., Schuëller, G.I.: A mode-based meta-model for the frequency response functions of uncertain structural systems. In: Bergen, B., De Munck, M., Desmet, W., Moens, D., Pluymers, B., Schuëller, G.I., Vandepitte, D. (eds.) *Leuven Symposium on Applied Mechanics in Engineering (LSAME 2008; CD-ROM)*, Katholieke Universiteit Leuven, Department of Mechanical Engineering Leuven, Belgium (2008)
10. Pichler, L., Pradlwarter, H.J., Schuëller, G.I.: A mode-based meta-model for the frequency response functions of uncertain structural systems. *Comput. Struct.* **87**(5–6), 332–341 (2009)
11. Pradlwarter, H.J., Schuëller, G.I.: Uncertain linear structural systems in dynamics: Efficient stochastic reliability assessment. *Comput. Struct.* doi:10.1016/j.compstruc.2009.06.010 (2009)
12. Pradlwarter, H.J., Schuëller, G.I., Schenk, C.A.: A computational procedure to estimate the stochastic dynamic response of large non-linear FE-Systems. *Comput. Method. Appl. Mech. Eng.* **192**(7–8), 777–801 (2002)
13. Schenk, C.A., Schuëller, G.I.: *Uncertainty Assessment of Large Finite Element Systems*. Springer, Berlin/Heidelberg/New York (2005)
14. Schuëller, G.I., Pradlwarter, H.J.: On the stochastic response of nonlinear FE-models. *Arch. Appl. Mech.* **69**, 765–784 (1999)
15. Schuëller, G.I., Pichler, L., Pradlwarter, H.J.: Meta-models of the eigensolution of uncertain structures. In: Brennan, M.J. (ed.) *7th European Conference on Structural Dynamics (EURODYN 2008; CD-ROM)*, Institute of Sound and Vibration, University of Southampton, Southampton (2008)
16. Soize, C.: Transient responses of dynamical systems with random uncertainties. *Probabilistic Eng. Mech.* **16**, 363–372 (2001)
17. Soize, C.: A comprehensive overview of a non-parametric probabilistic approach of model uncertainties for predictive models in structural dynamics. *J. Sound Vib.* **288**(3), 623–652 (2005)

Nonlinear Dynamic Response Variability and Reliability of Frames with Stochastic Non-Gaussian Parameters

George Stefanou and Michalis Fragiadakis

Abstract Current research efforts for the efficient prediction of the dynamic response of structures with parameter uncertainty concentrate on the development of new and the improvement of existing methods. However, they are usually limited to linear elastic analysis considering only monotonic loading. In order to investigate realistic problems of structures subjected to transient seismic actions, a novel approach has been recently introduced by the authors. This approach is used here to assess the nonlinear stochastic response and reliability of a three-storey steel moment-resisting frame in the framework of Monte Carlo simulation (MCS) and translation process theory. The structure is modeled with a mixed fiber-based, beam-column element, whose kinematics is based on the natural mode method. The adopted formulation leads to the reduction of the computational cost required for the calculation of the element stiffness matrix, while increased accuracy compared to traditional displacement-based elements is achieved. The uncertain parameters of the problem are the Young modulus and the yield stress, both described by homogeneous non-Gaussian translation stochastic fields that vary along the element. The frame is subjected to natural seismic records that correspond to three levels of increasing seismic intensity. Under the assumption of a pre-specified power spectral density function of the stochastic fields that describe the two uncertain parameters, the response variability of the frame is computed using MCS. Moreover, a parametric investigation is carried out providing useful conclusions regarding the influence of the correlation length of the stochastic fields on the response variability and reliability of the frame.

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1 Introduction

In the last few years, the problems of dynamic response analysis and reliability assessment of structures with uncertain system and excitation parameters have been the subject of extensive research. The case of deterministic and stochastic linear systems subjected to random excitation has been studied first and is well represented in the literature [1–3]. An approximate method for the response variability calculation of dynamical systems with uncertain stiffness and damping ratio can be found in [1]. This approach is based on complex mode analysis where the variability of each mode is analyzed separately and can efficiently treat a variety of probability distributions assumed for the system parameters. The time-varying reliability evaluation of uncertain structures subjected to stochastic earthquake ground motion is treated in [2] using conditional crossing rate estimation and the perturbation-based stochastic finite element method. Recently, an exact non-statistical method has been proposed for the dynamic analysis of FE-discretized uncertain linear structures in the frequency domain [3].

In contrast to the linear case, the efficient prediction of the nonlinear dynamic response of structures with uncertain system properties still poses a major challenge in the field of computational stochastic mechanics [4–6]. This can be explained by the fact that most of the methods developed for the analysis of linear systems are inefficient or inappropriate for the nonlinear case. For example, the analysis of uncertain nonlinear systems is generally not feasible using frequency domain analysis techniques [7]. The existing methods for response statistics calculation in this case are mostly based on simulation [4, 8] or on the perturbation approach [9]. Applications of the response surface method have also been proposed [10], while studies can be found on the statistical equivalent linearization (EQL) method for the response variability and reliability estimation of discrete nonlinear systems [11]. Alternatively, a probability density evolution method (PDEM) has been developed for this purpose [12] and a neural network-based approach has been proposed for the efficient fragility assessment of steel frames with uncertain material properties modeled as normal random variables [13]. It is worth noting that the theory of non-Gaussian translation processes (used here for the uncertain system properties) has also been applied directly to the reliability analysis of dynamic systems under limited information. This method delivers accurate results for the case of linear and nonlinear dynamic systems assuming stationary output but can be easily extended to a special class of non-stationary, non-ergodic output [14].

It is well known that the main drawback of the perturbation approach is the significant loss of accuracy when the level of uncertainty of the system properties is high. On the other hand, the computational effort required by statistical approaches such as Monte Carlo simulation (MCS) for the analysis of large-scale structures is considerable thus making essential the use of efficient solution strategies and parallel processing [6]. In addition, the validity of EQL is questionable in some cases and the method may even produce misleading results [15]. Based on the aforementioned observations, it is concluded that an approach combining accuracy and computational efficiency is still very desirable in this area.

In order to investigate realistic problems of structures subjected to seismic loading, a novel approach, combining MCS and nonlinear response history analysis with the stochastic field theory, has been recently introduced [16]. The proposed methodology is used here to assess the response variability and reliability of a benchmark three-storey steel moment-resisting frame [17]. The structure is modeled with a mixed fiber-based, beam-column element, whose kinematics is based on the natural mode method. The adopted formulation leads to the reduction of the computational cost required for the computation of the element stiffness matrix, while increased accuracy compared to traditional displacement-based elements is achieved [18, 19]. The uncertain parameters are the Young modulus and the yield stress, both described by homogeneous non-Gaussian translation stochastic fields [20]. The frame is subjected to natural seismic records that correspond to three levels of increasing seismic intensity. Under the assumption of a pre-specified power spectral density function of the stochastic fields describing the two uncertain parameters, the response variability and reliability of the frame is computed using MCS. Finally, a parametric investigation is carried out providing useful conclusions regarding the influence of the spectral characteristics of the stochastic fields on the response variability.

2 Force-Based Formulation of the Beam-Column Element

Inelastic analysis of frame structures can be performed either with a lumped or with a distributed plasticity formulation. Distributed plasticity elements are considered more accurate and, in general, are distinguished to displacement-based and to force-based elements. The latter approach, also known as flexibility formulation, has a number of distinct features over the former, especially if it is adopted in the framework of a mixed beam-column formulation [21]. The force-based formulation requires a single beam-column element per member to simulate its material nonlinear response, since it uses force interpolation functions. Consequently, the element equilibrium is always satisfied, while compatibility of deformations is satisfied by integrating the section deformations to obtain the element deformations and the nodal displacements. In order to numerically calculate the stiffness matrix, a number of sections along the beam-column element are chosen, while every section is divided to a number of monitoring sections, known as fibers. Fibers are simply integration points of a low order quadrature at the section level and are used to evaluate the section stiffness as follows:

$$\mathbf{D}_{\text{sec}} = \mathbf{k}_{\text{sec}} \mathbf{d}_{\text{sec}} \Leftrightarrow \mathbf{D}_{\text{sec}} = \left(\int_A \frac{\partial \sigma}{\partial \varepsilon} \begin{bmatrix} 1 & -y \\ -y & y^2 \end{bmatrix} dA \right) \mathbf{d}_{\text{sec}} \quad (1)$$

where y is the distance of a fiber from the neutral axis, $\mathbf{D}_{\text{sec}}$ are the section forces and $\mathbf{d}_{\text{sec}} = [\varepsilon_x, \kappa]^T$ is the vector of section deformations that consists of the axial strain ε_x and the curvature κ . If the response is linear elastic, the diagonal terms

of the section stiffness matrix become equal to EA and EI , respectively, while the off-diagonal terms are zero. If the section flexibility matrix is $\mathbf{f}_{\text{sec}} = \mathbf{k}_{\text{sec}}^{-1}$, the element flexibility matrix $\mathbf{F} = \mathbf{K}^{-1}$ is obtained as follows:

$$\mathbf{F} = \mathbf{K}^{-1} = \int_{-1}^1 \mathbf{b}^T \mathbf{k}_{\text{sec}}^{-1} \mathbf{b} d\xi = \int_{-1}^1 \mathbf{b}^T \mathbf{f}_{\text{sec}} \mathbf{b} d\xi = \sum_{i=1}^{NP} w_i \mathbf{b}^T(\xi_i) \mathbf{f}_{\text{sec}}(\xi_i) \mathbf{b}(\xi_i) \quad (2)$$

The above equation implies that numerical integration is required in order to obtain the element flexibility matrix, where NP is the number of integration points along the element. In force-based elements, the Gauss–Lobatto quadrature is preferred because it considers as sections of integration the beam ends where the bending moment is maximum (provided that no other element loads are present). This integration scheme requires at least three integration sections, while typically four to six sections are chosen.

The kinematics of the element used in this study follow the principles of the natural mode method proposed by Argyris [22]. According to the natural mode method, the displacement field can be decomposed into three rigid body modes $\boldsymbol{\rho}_0$ and three straining modes $\boldsymbol{\rho}_N$ shown in Fig. 1. In a flexibility-based element the calculation of the natural element forces is performed iteratively for each element. The first step of the iterative procedure is to determine the vector of the natural forces. Then using force interpolation functions, the section forces are obtained and subsequently they are corrected according to the constitutive law. From the corrected forces the section deformations are obtained using Eq. 1 and are then integrated according to

$$\boldsymbol{\rho}_N = \int_{-1}^1 \mathbf{b}^T(\xi) \mathbf{d}_{\text{sec}} d\xi \quad (3)$$

in order to obtain the residual natural modes. $\mathbf{b}$ is the interpolation matrix, which is a function of the natural coordinate $\xi \in [-1, 1]$ along the element. The iterative process in the element level is terminated when an energy convergence criterion is satisfied.

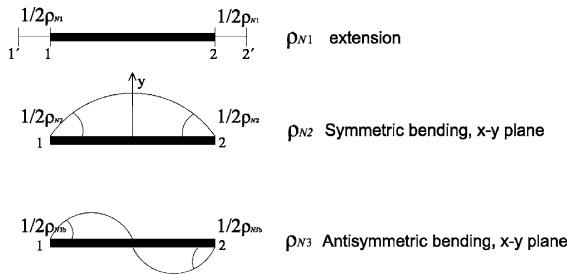


Fig. 1 Natural straining modes in the x - y plane

3 Stochastic Stiffness Matrix

In the context of stochastic finite element analysis, the uncertain system properties are usually represented by stochastic fields [23]. The statistical properties of these fields are based either on experimental measurements or, when no experimental results are available, on an assumed variation. In this work, the Young modulus E and the yield stress σ_y of the structure are assumed to be described by two uncorrelated 1D-1V homogeneous non-Gaussian stochastic fields:

$$E(x) = E_0 [1 + f_1(x)] \quad (4)$$

$$\sigma_y(x) = \sigma_{y0} [1 + f_2(x)] \quad (5)$$

where E_0 is the mean value of the Young modulus, σ_{y0} is the mean value of the yield stress of the material and $f_1(x)$, $f_2(x)$ are two zero-mean non-Gaussian homogeneous stochastic fields corresponding to the variability of the Young modulus and the yield stress, respectively.

As the entries of the element flexibility matrix $\mathbf{F}$ of Eq. 2 are nonlinear functions of the uncertain material properties, it is not possible to establish a closed form expression for the stochastic flexibility matrix. However, the analytical expression of the stochastic section stiffness matrix $\mathbf{k}_{\text{sec}}$ with stochastic material properties based on Eqs. 4 and 5 has been derived in [16]. The stochastic flexibility matrix of the beam-column element is calculated numerically using its deterministic formulation given in Eq. 2 and the stochastic stiffness.

4 Simulation of Uncertain Parameters Using Non-Gaussian Translation Fields

In this work, a non-Gaussian assumption is made for the distribution of the uncertain parameters of the frame. This choice is in accordance to the fact that several quantities arising in practical engineering problems (e.g. material and geometric properties of structural systems, soil properties, wind loads, waves) are found to exhibit non-Gaussian probabilistic characteristics. In addition, the non-Gaussian assumption permits to efficiently treat the case of large input variability without violating the physical constraints of the material properties.

A number of studies in the literature have been focused on producing a realistic definition of a non-Gaussian sample function from a simple transformation of an underlying Gaussian field with known second-order statistics. Thus, if $g(x)$ is a homogeneous zero-mean Gaussian field with unit variance and spectral density function (SDF) $S_{gg}(\kappa)$ (or equivalently autocorrelation function $R_{gg}(\xi)$), a homogeneous non-Gaussian stochastic field $f(x)$ with power spectrum $S_{ff}^T(\kappa)$ is defined as:

$$f(x) = F^{-1} \cdot \Phi[g(x)] \quad (6)$$

where Φ is the standard Gaussian cumulative distribution function and F is the non-Gaussian marginal cumulative distribution function (CDF) of $f(x)$. The transform $F^{-1} \cdot \Phi$ is a memory-less translation since the value of $f(x)$ at an arbitrary point x depends on the value of $g(x)$ at the same point only and the resulting non-Gaussian field is called a translation field [20].

Translation fields have a number of useful properties such as the analytical calculation of crossing rates and extreme value distributions. They also have some shortcomings, the most important of which from a practical point of view is the possible incompatibility between their marginal distribution F and correlation structure $S_{ff}^T(\kappa)$. F and $S_{ff}^T(\kappa)$ (or $R_{ff}^T(\xi)$) of a translation field have to satisfy a specific compatibility condition derived directly from the definition of its autocorrelation function [20]. If these two quantities are proven to be incompatible i.e. if $R_{ff}^T(\xi)$ has certain values lying outside a range of admissible values and/or $R_{gg}(\xi)$ is not positive definite and therefore not admissible as an autocorrelation function, there is no translation field with the prescribed characteristics. In this case, one has to resort to translation fields that match the target marginal distribution and/or SDF approximately. Since experimental data can lead to a theoretically incompatible pair of F and $S_{ff}^T(\kappa)$, iterative algorithms have been recently developed, which extend the translation field concept and lead to the generation of non-Gaussian fields having the prescribed characteristics [24–26]. It is worth noting that the theory of non-Gaussian translation fields has been recently applied directly to the reliability analysis of linear and nonlinear dynamic systems under limited information [14].

In the present work, Eq. 6 is used for the generation of non-Gaussian translation sample functions representing the uncertain parameters of the problem. Sample functions of the underlying Gaussian field $g(x)$ are generated using the spectral representation method [27]. The SDF $S_{gg}(\kappa)$ of $g(x)$ used in the numerical example is assumed to correspond to an autocorrelation function of square exponential type and is given by:

$$S_{gg}(\kappa) = \frac{\sigma_g^2 b}{2\sqrt{\pi}} \exp\left(-\frac{b^2 \kappa^2}{4}\right) \quad (7)$$

where σ_g is the standard deviation of the stochastic field and b denotes the parameter that influences the shape of the spectrum and is proportional to the correlation length of the stochastic field along the x -axis. The SDF of the translation field obtained from Eq. 6 will be slightly different from $S_{gg}(\kappa)$ (see e.g. [28]).

Using the procedure described in this section, a large number N_{SAMP} of non-Gaussian sample functions are produced, leading to the generation of a set of stochastic stiffness matrices. The associated structural problem is solved N_{SAMP} times and the response variability and reliability can finally be calculated by obtaining the statistics of the N_{SAMP} simulations.

5 Numerical Examples

In this section, the three-storey steel moment-resisting frame shown in Fig. 2 is used for a numerical implementation of the above described methodology. The frame has been designed for a Los Angeles site, following the 1997 NEHRP (National Earthquake Hazard Reduction Program) provisions in the framework of the SAC/FEMA program [29]. The dynamic response of the building is dominated by the fundamental mode which has a period value equal to $T_1 = 1.02$ s when the mean value of the modulus of elasticity is used. All response history analyses were performed using a force-based, beam-column fiber element with five integration sections implemented on a general purpose finite element program [30]. Geometric nonlinearities were not considered in the analysis. Rayleigh damping is used to obtain a damping ratio of 2% for the first and the fourth mode. The material law is considered to be bilinear with pure kinematic hardening, where the properties of each integration section differ according to the stochastic fields of Eqs. 4 and 5. The frame section properties are given in Table 1. The gravity loading applied is 32.22 kN/m for the first two stories and 28.76 kN/m for the top storey. These values are used also to obtain the nodal masses resulting to a lumped mass matrix.

Three sets of five strong ground motion records are used as input to the nonlinear dynamic procedure (Table 2). The three sets correspond to three levels of increasing hazard: low, medium and high (Fig. 3). Although the predominant frequencies of the records lie at period values less than the fundamental elastic period of the frame, it is clear that for many of the records the amplification at the vicinity of 1.0 s is rather significant. The records chosen differ in terms of amplitude, frequency content, duration, etc. and therefore this variability is expected to be transferred

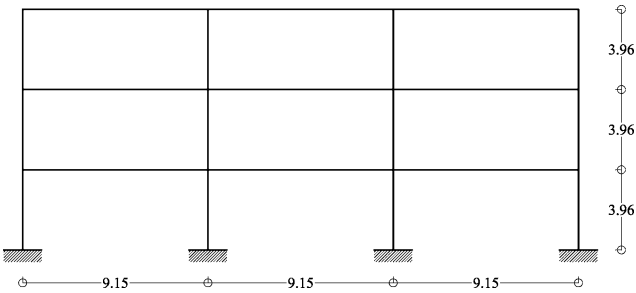


Fig. 2 The three-storey LA3 steel frame

Table 1 The frame section properties

Storey	Beams	Columns	
		Exterior	Interior
1	W 30 × 118	W 14 × 257	W 14 × 311
2	W 30 × 116	W 14 × 257	W 14 × 311
3	W 24 × 68	W 14 × 257	W 14 × 311

Table 2 The ground motion records used

ID (level)	Earthquake	φ^{oa}	Mw ^b	R ^c	Soil ^d	PGA ^e	$S_a(T_1)^e$
1 (1/1)	Imperial Valley, 1979	045	6.5	31.7	C, D	0.042	0.0214
2 (2/1)	Imperial Valley, 1979	135	6.5	31.7	C, D	0.057	0.0599
3 (3/1)	Imperial Valley, 1979	180	6.5	15.1	C, D	0.11	0.0959
4 (4/1)	Imperial Valley, 1979	090	6.5	15.1	C, D	0.074	0.0923
5 (5/1)	Imperial Valley, 1979	285	6.5	32.6	C, D	0.147	0.0546
6 (1/2)	Northridge, 1994	090	6.7	31.3	B, B	0.239	0.1615
7 (2/2)	Imperial Valley, 1979	090	6.5	31.7	C, D	0.057	0.2476
8 (3/2)	Loma Prieta, 1989	270	6.9	28.8	C, D	0.207	0.2596
9 (4/2)	Superstition Hills, 1987	090	6.7	24.4	C, D	0.18	0.2038
10 (5/2)	Loma Prieta, 1989	360	6.9	28.8	C, D	0.209	0.2882
11 (1/3)	Superstition Hills, 1987	360	6.7	24.4	C, D	0.2	0.4417
12 (2/3)	Northridge, 1994	360	6.7	25.5	C, D	0.358	0.4532
13 (3/3)	Loma Prieta, 1989	000	6.9	28.8	-, D	0.371	1.0079
14 (4/3)	Loma Prieta, 1989	000	6.9	16.9	-, D	0.370	0.5368
15 (5/3)	Loma Prieta, 1989	090	6.9	16.9	-, D	0.638	0.5787

^aComponent.

^bMoment magnitude.

^cClosest distance to fault rupture.

^dUSGS, Geomatrix soil class.

^eIn units of g.

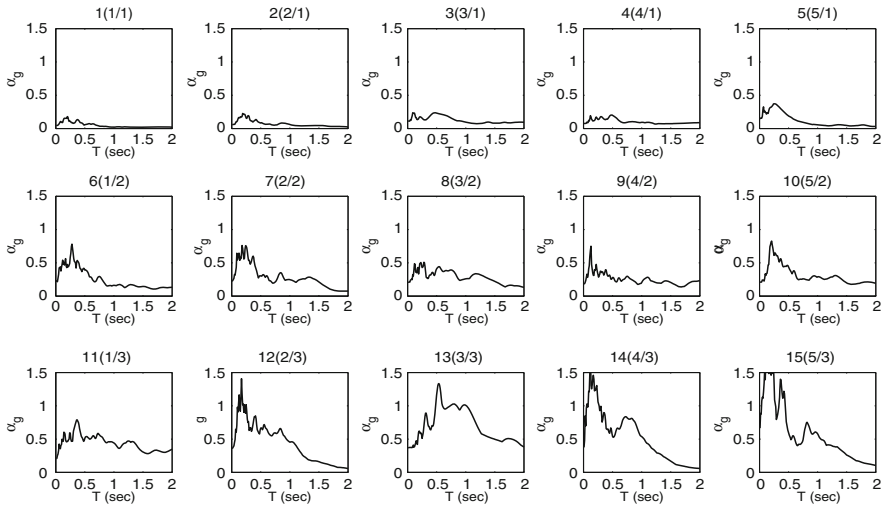


Fig. 3 5% damped response spectra of the 15 natural ground motion records of Table 2

to the statistics of the analysis, producing significant record-to-record variability. The 15 natural records are used as input to the analysis in order to compute the mean response quantities, the dispersion and the reliability of the frame for the three intensity levels.

5.1 Response Variability and Reliability of the Frame

The spatial variability in Young modulus and yield stress of the frame is described by two uncorrelated 1D-1V homogeneous non-Gaussian translation stochastic fields with zero mean and coefficient of variation (COV) equal to 0.10. A slightly skewed shifted lognormal distribution defined in the range $[-1, +\infty]$ is assumed for the two stochastic fields. The skewness of the lognormal distribution is equal to 0.30. E and σ_y are simultaneously varying in all the cases examined. The representative response quantity whose statistics are monitored is the maximum interstorey drift, which for brevity will be simply referred to as drift and denoted as $\theta_{\max}$. This parameter is a well-known engineering demand parameter (EDP) that captures the seismic demand and its distributions along the height of the structure. The response statistics have been calculated using 1,000 Monte Carlo simulations. This number of simulations represents sufficiently well the prescribed first two moments of the stochastic fields, while as shown in Fig. 4, statistical convergence is practically achieved after 400 simulations. This trend was observed for all ground motions considered.

The sensitivity of $\theta_{\max}$ with respect to the scale of correlation of the stochastic fields, quantified with the aid of the correlation length parameter b of the underlying Gaussian field, is examined for the ground motions of the three sets. For this purpose, several sets of sample functions of E and σ_y are generated using Eq. 6 each for a different value of parameter b . Six representative values of b varying from weak to strong correlation are considered ($b = 0.2, 1.0, 2.0, 10, 20$ and 100).

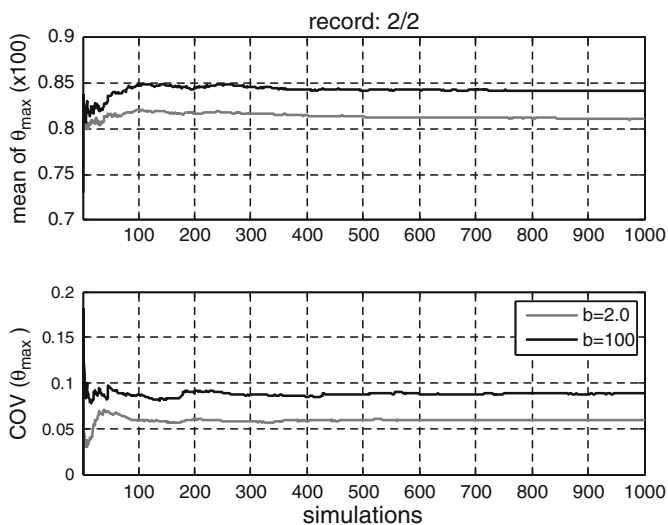


Fig. 4 Statistical convergence of mean and COV of maximum interstorey drift $\theta_{\max}$ – lognormal distribution of E , σ_y with COV = 0.1

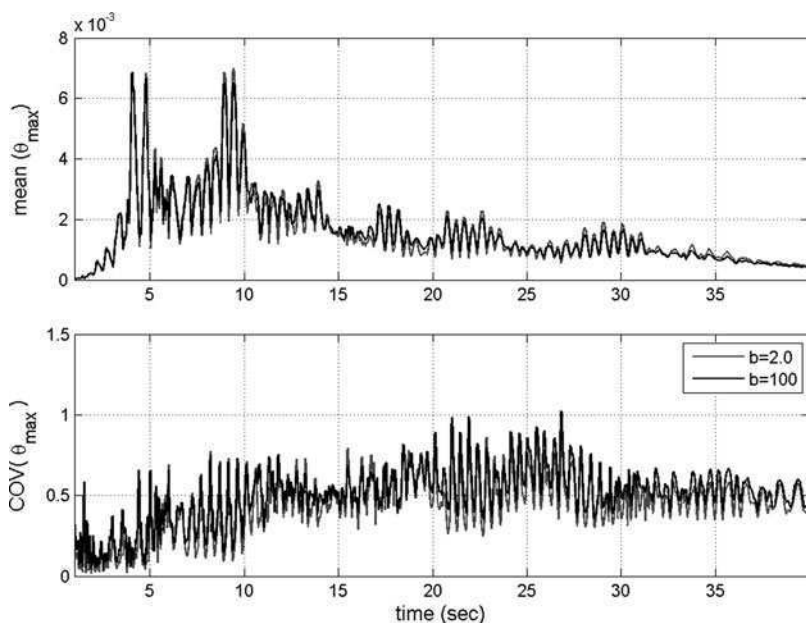


Fig. 5 Time histories of mean, $\text{COV}(\theta_{\max})$ – lognormal distribution of E , σ_y for record 5/2. The time histories are shown for $b = 2.0$ and 100

The dynamic response of the frame is highly non-stationary as it can be seen in Fig. 5 where the evolution with time of the mean and the COV of $\theta_{\max}$ are depicted for two correlation length values and for a record of medium intensity. An important observation can be made regarding the variability of $\theta_{\max}$. In contrast to the static case where the displacement variability shows always the same trend, starting from small values for small correlation lengths corresponding to white noise stochastic fields up to large values for large correlation lengths [23], the COV of $\theta_{\max}$ varies significantly not only with the correlation length b but also in different ways among the records of the same intensity level (Fig. 6b). In some cases, the effect of b becomes negligible and then the record-to-record variability is predominant (e.g. records 1/1, 5/1 and 3/3). In addition, a large magnification of uncertainty is observed in some cases, which is more pronounced for records 1/1, 4/1, 4/2 and 5/2, where the response COV tends to values that are 1.4–1.8 times greater than the corresponding input COV(= 0.1). When stochastic earthquake loading is considered in addition to uncertain system properties, the magnification of uncertainty becomes even more pronounced [16]. In contrast, the mean value of drift, although presenting an important record-to-record variability, is practically not affected by the correlation length parameter b (see Fig. 6a).

Similar results have been obtained using an L-beta distribution for the material properties [16]. The small difference between the results obtained with the lognormal and L-beta distributions can be explained by the small value of

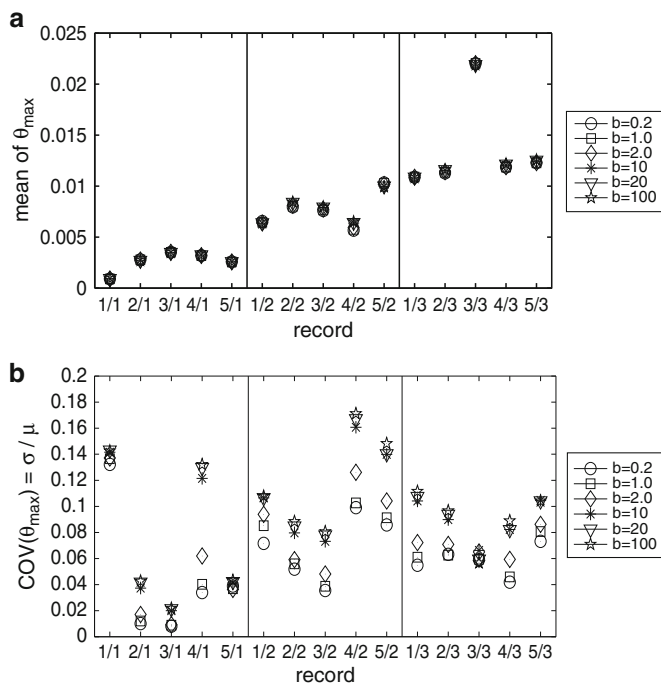


Fig. 6 (a) Mean and (b) $\text{COV}(\theta_{\max})$ for different values of correlation length parameter b and the 15 natural ground motions of Table 2

$\text{COV}(= 0.1)$ of the system properties. This is why a Gaussian distribution is often used in analogous cases in the literature without introducing large errors. However, it has been shown in [31] that the variance of the response of a system with a Gaussian stiffness is infinite. Since this is not physically sound, a stochastic stiffness should not be assumed to be Gaussian. In addition, the non-Gaussian assumption permits to efficiently treat the case of large input COV appearing in many engineering problems e.g. in soil mechanics [32]. The skewness of $\theta_{\max}$ has been also calculated and is depicted in Fig. 7 where a significant influence of the correlation length parameter b and an important record-to-record variability can also be observed. The values of skewness obtained in each record and seismic intensity are substantially different from the skewness of the lognormal distribution describing the material properties thus underlying the strong non-linearity of the problem that causes a significant change in the statistical moments (and distribution) of the response.

Using the results obtained in this section, the reliability of the frame can finally be calculated. Figure 8 shows the CDF of $\theta_{\max}$ of record 4/2 for two different values of $b = 1, 100$. If the reliability of the frame is defined as the maximum interstorey drift not exceeding a threshold e.g. 6.5×10^{-3} , the reliability can be obtained from Fig. 8 for both cases of b as 0.915 and 0.555, respectively. It is worth noting that the

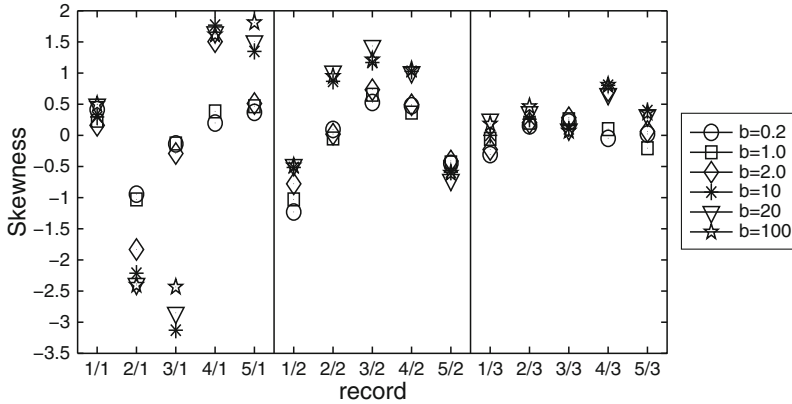


Fig. 7 Skewness of $\theta_{\max}$ for different values of correlation length parameter b and the 15 natural ground motions of Table 2

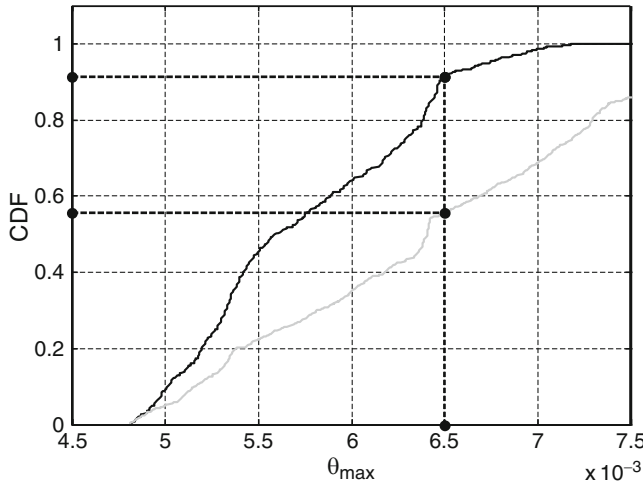


Fig. 8 CDF of $\theta_{\max}$ of record 4/2 for correlation length parameter $b = 1$ (black line) and $b = 100$ (grey line)

reliability is substantially smaller in the second case. Figure 9 shows the probability density function (PDF) of $\theta_{\max}$ of record 4/2 for the same two values of b , computed using the kernel density estimation method [33]. Simultaneously shown are the normal and lognormal distributions with a mean and standard deviation identical to those of the computed PDF, and the extreme value distribution with the same mean as that of the computed PDF. It can be observed that these widely adopted probability distributions are quite different from the real PDF of the response, which clearly has a bimodal form especially in the case of small correlation length.

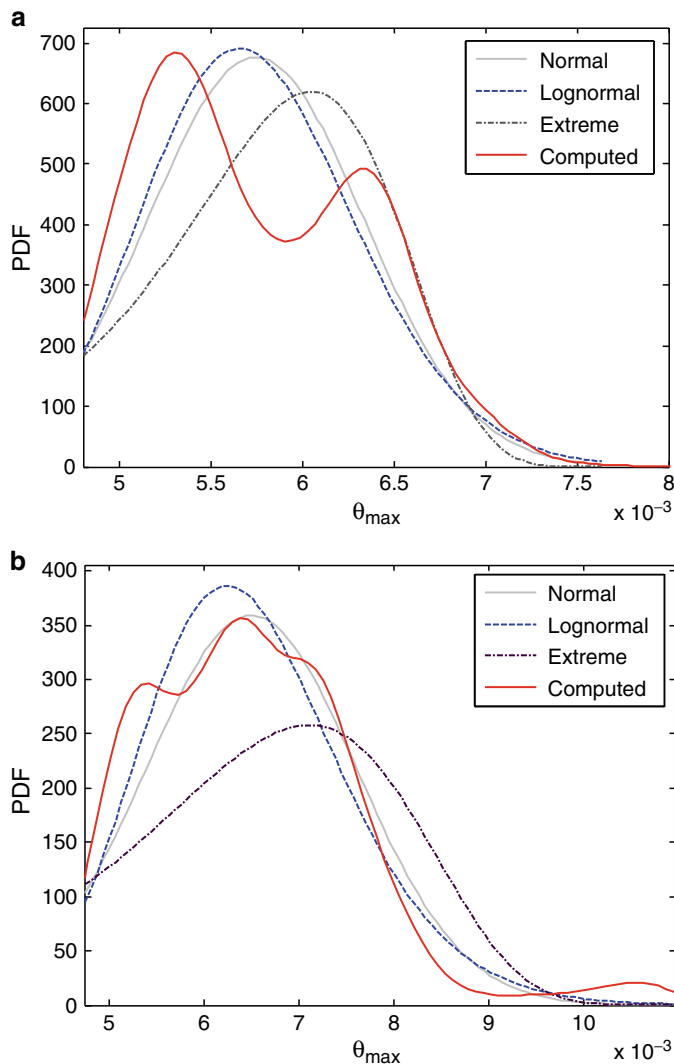


Fig. 9 PDF of $\theta_{\max}$ of record 4/2 for correlation length parameter (a) $b = 1$ and (b) $b = 100$

6 Conclusions

A stochastic response history and reliability analysis of a steel frame having uncertain non-Gaussian material parameters and subjected to seismic loading has been performed. The frame is modeled with a mixed fiber-based, beam-column element, whose kinematics are based on the natural mode method. The adopted formulation provides increased accuracy compared to traditional displacement-based

elements and offers significant computational advantages for the analysis of systems with stochastic properties. Two uncorrelated 1D-1V homogeneous non-Gaussian translation stochastic fields with prescribed marginal CDF and SDF have been used for the description of the random spatial fluctuation of the material properties. The variability of the maximum interstorey drift $\theta_{\max}$ and the reliability of the frame have been computed using MCS.

A parametric investigation revealed the significant influence of the scale of correlation of the stochastic fields (quantified via the correlation length parameter b) and of the different seismic records on the response variability: the COV and skewness of $\theta_{\max}$ have been found to vary quantitatively with b and in many different ways between the records of the same intensity level. Finally, a large magnification of uncertainty has been observed in some cases, leading to response COV values that were 1.4–1.8 times greater than those of the input COV. This magnification of uncertainty can be even more pronounced when stochastic earthquake loading is considered in addition to uncertain system properties. These observations underline the importance of a realistic uncertainty quantification and propagation in nonlinear dynamic analysis of engineering systems.

References

1. Papadimitriou, C., Katafygiotis, L.S., Beck, J.L.: Approximate analysis of response variability of uncertain linear systems. *Probab. Eng. Mech.* **10**, 251–264 (1995)
2. Chaudhuri, A., Chakraborty, S.: Reliability of linear structures with parameter uncertainty under non-stationary earthquake. *Struct. Safe* **28**, 231–246 (2006)
3. Falsone, G., Ferro, G.: An exact solution for the static and dynamic analysis of FE discretized uncertain structures. *Comput. Methods Appl. Mech. Eng.* **196**, 2390–2400 (2007)
4. Schuëller, G.I., Pradlwarter, H.J.: On the stochastic response of nonlinear FE models. *Arch. Appl. Mech.* **69**, 765–784 (1999)
5. Manolis, G.D., Koliopoulos, P.K.: *Stochastic Structural Dynamics in Earthquake Engineering*. WIT Press, UK (2001)
6. Johnson, E.A., Proppe, C., Spencer Jr., B.F., Bergman, L.A., Székely, G.S., Schuëller, G.I.: Parallel processing in computational stochastic dynamics. *Probab. Eng. Mech.* **18**, 37–60 (2003)
7. Iwan, W.D., Huang, C.T.: On the dynamic response of nonlinear systems with parameter uncertainties. *Int. J. Nonlinear Mech.* **31**, 631–645 (1996)
8. Muscolino, G., Ricciardi, G., Cacciola, P.: Monte Carlo simulation in the stochastic analysis of nonlinear systems under external stationary Poisson white noise input. *Int. J. Nonlinear Mech.* **38**, 1269–1283 (2003)
9. Liu, W.K., Belytschko, T., Mani, A.: Probability finite elements for nonlinear structural dynamics. *Comput. Methods Appl. Mech. Eng.* **56**, 61–81 (1986)
10. Huh, J., Haldar, A.: Stochastic finite element-based seismic risk of nonlinear structures. *J. Struct. Eng. (ASCE)* **127**, 323–329 (2001)
11. Proppe, C., Pradlwarter, H.J., Schuëller, G.I.: Equivalent linearization and Monte Carlo simulation in stochastic dynamics. *Probab. Eng. Mech.* **18**, 1–15 (2003)
12. Li, J., Chen, J.B.: The probability density evolution method for dynamic response analysis of nonlinear stochastic structures. *Int. J. Numer. Methods Eng.* **65**, 882–903 (2006)
13. Lagaros, N.D., Fragiadakis, M.: Fragility assessment of steel frames using neural networks. *Earthquake Spectra* **23**, 735–752 (2007)
14. Field Jr., R.V., Grigoriu, M.: Reliability of dynamic systems under limited information. *Probab. Eng. Mech.* **24**, 16–26 (2009)

15. Bernard, P.: Stochastic linearization: what is available and what is not. *Comput. Struct.* **67**, 9–18 (1998)
16. Stefanou, G., Fragiadakis, M.: Nonlinear dynamic analysis of frames with stochastic non-Gaussian material properties. *Eng. Struct.* **31**, 1841–1850 (2009)
17. Gupta, A., Krawinkler, H.: Behavior of ductile SMRFs at various seismic hazard levels. *J. Struct. Eng. (ASCE)* **126**, 98–107 (2000)
18. Papaioannou, I., Fragiadakis, M., Papadrakakis, M. Inelastic analysis of framed structures using the fiber approach. In: *Proceedings of the 5th International Congress on Computational Mechanics (GRACM 05)*, vol. I. Limassol, Cyprus, 29 June–1 July 2005, pp. 231–238
19. Barbato, M., Conte, J.P.: Finite element response sensitivity analysis: a comparison between force-based and displacement-based frame element models. *Comput. Methods Appl. Mech. Eng.* **194**, 1479–1512 (2005)
20. Grigoriu, M.: Simulation of stationary non-Gaussian translation processes. *J. Eng. Mech. (ASCE)* **124**, 121–126 (1998)
21. Spacone, E., Filippou, F.C., Taucer, F.F.: Fiber beam-column model for non-linear analysis of R/C frames: Part I Formulation. *Earthquake Eng. Struct. Dyn.* **25**, 711–725 (1996)
22. Argyris, J., Tenek, L., Mattsson, A.: BEC: a 2-node fast converging shear-deformable isotropic and composite beam element based on 6 rigid-body and 6 straining modes. *Comput. Methods Appl. Mech. Eng.* **152**, 281–336 (1988)
23. Stefanou, G., Papadrakakis, M.: Stochastic finite element analysis of shells with combined random material and geometric properties. *Comput. Methods Appl. Mech. Eng.* **193**, 139–160 (2004)
24. Deodatis, G., Micaletti, R.C.: Simulation of highly skewed non-Gaussian stochastic processes. *J. Eng. Mech. (ASCE)* **127**, 1284–1295 (2001)
25. Lagaros, N.D., Stefanou, G., Papadrakakis, M.: An enhanced hybrid method for the simulation of highly skewed non-Gaussian stochastic fields. *Comput. Methods Appl. Mech. Eng.* **194**, 4824–4844 (2005)
26. Bocchini, P., Deodatis, G.: Critical review and latest developments of a class of simulation algorithms for strongly non-Gaussian random fields. *Probab. Eng. Mech.* **23**, 393–407 (2008)
27. Shinozuka, M., Deodatis, G.: Simulation of stochastic processes by spectral representation. *Appl. Mech. Rev. (ASME)* **44**, 191–203 (1991)
28. Papadopoulos, V., Stefanou, G., Papadrakakis, M.: Buckling analysis of imperfect shells with stochastic non-Gaussian material and thickness properties. *Int. J. Solids Struct.* **46**, 2800–2808 (2009)
29. SAC: State of the Art Report on System Performance of Steel Moment Frames Subjected to Earthquake Ground Shaking. FEMA-355C, Federal Emergency Management Agency, Washington, DC (2000)
30. Taylor, R.L.: “FEAP: A Finite Element Analysis Program,” User Manual, Version , Department of Civil and Environmental Engineering, University of California at Berkeley, Berkeley, CA, [http://www.ce.berkeley.edu/~sim\\$rlt/feap/](http://www.ce.berkeley.edu/~sim$rlt/feap/) (2000)
31. Schevenels, M., Lombaert, G., Degrande, G. (2004) Application of the stochastic finite element method for Gaussian and non-Gaussian systems. In: *Proceedings of ISMA 2004 Conference*, Leuven, Belgium, September 20–22, 2004, pp. 3299–3313
32. Fenton, G.A., Griffiths, G.V.: Bearing-capacity prediction of spatially random c - ϕ soils. *Can. Geotech. J.* **40**, 54–65 (2003)
33. Bowman, A.W., Azzalini, A.: *Applied Smoothing Techniques for Data Analysis*. Oxford University Press, UK (1997)

The Role of Uncertainties in Aeolian Risk Assessment

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Abstract The design of slender structures subject to wind actions must take into account many sources of uncertainty in the environmental parameters that describe the wind field at the site, in the structural mechanical properties, and in the parameters that influence the interaction phenomena. For the sake of simplicity, three types of uncertainty can be distinguished: (i) aleatory uncertainty, related to the natural variability of the parameters and their unpredictability; (ii) epistemic uncertainty, related both to the lack of information and to the errors of experimental measures; (iii) model uncertainty, related to the choice of the models of wind actions, structural dynamic response and interaction phenomena. In this paper, the relevance of the different sources of uncertainty is discussed and illustrated with reference to an example case. In detail, the effects on the risk assessment of a long span suspension bridge induced by the uncertainty of the model of the aeroelastic forces are investigated. The study is aimed at the development of a general framework for Performance-Based Wind Engineering.

1 Introduction

In the last decades, the number of structures sensitive to wind actions has increased in parallel with the progress of construction technologies, and the assessment of the Aeolian risk has become essential [1]. As outlined more than 20 years ago [2], probabilistic approaches have to be adopted to deal with the stochastic nature of wind actions and structural response. Several sources of uncertainty have to be considered: they can be roughly distinguished in three groups, that concern respectively (i) the probabilistic characterization of the basic wind parameters at the site (mean wind velocity, its occurrence rate and direction, turbulence intensity, etc.), (ii) the

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structural (static and dynamic) properties, and (iii) the parameters modelling the wind-structure interaction phenomena (aerodynamic coefficients, flutter derivatives, Strouhal number, etc.).

While the first two sources of uncertainty are encountered also in other fields (e.g. earthquake, fire, blast and flood engineering), the uncertainties related to interaction parameters is specific of wind engineering and have a crucial relevance in case of slender structures whose motion is coupled with the air flow. Moreover, the dispersion of the values of the interaction parameters that can be found in the technical literature is usually high; in fact the nominal values that are derived by experimental tests [3] or numerical analyses are highly sensitive to, for example, the scale of wind tunnel physical models and the accuracy of experimental measures and/or numerical calculations.

In this paper a classification of the main sources of uncertainties in wind engineering is presented and the relevance of some of them is discussed, in order to improve the general framework for Performance-Based Wind Engineering (PBWE) that is currently under development [4, 5]. Long-span suspension bridges are structures very sensitive to wind, because of their high flexibility, low structural damping and comparatively little mass: consequently, wind-structure interaction phenomena are particularly relevant. Therefore, as an example case the Aeolian risk assessment of a long span suspension bridge is dealt with.

2 Sources of Uncertainty in Wind Engineering

In assessing the Aeolian risk of structures, different sources of uncertainty can be distinguished by considering the free-field conditions and the presence of the structure (Fig. 1).

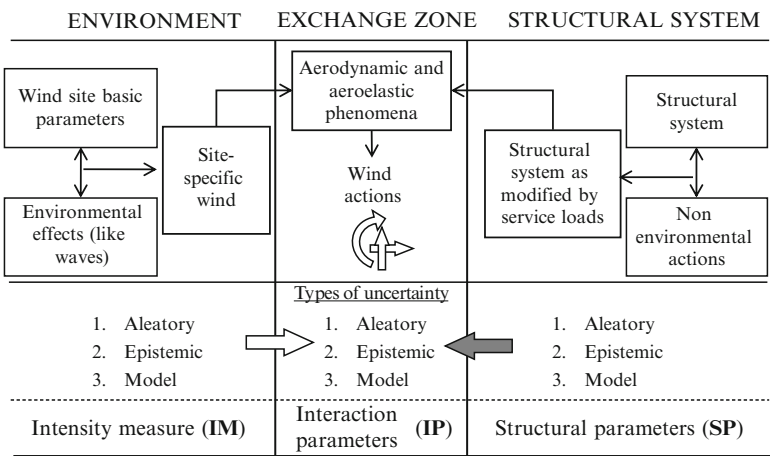


Fig. 1 Sources of uncertainty in wind engineering

By definition, in the *environment* the wind field is evaluated as if the structure were absent (free-field wind), and the basic site-dependent parameters of the wind field (mean value of the velocity in each direction, turbulence intensity, dominant direction of the strong winds, etc.) are not influenced by the presence of the structure, while they can be affected by the interaction with other environmental agents [6]: a typical example is the interaction between wind and waves in offshore sites. The *exchange zone* is the region around the structure where the structure and the wind field are strongly related, and the effects of wind-structure interaction, as well as the presence of nearby structures, cannot be disregarded: aerodynamic and aeroelastic phenomena are essential in determining the relevant features of the response to wind actions. Non environmental actions can influence the structural response by modifying the aerodynamic and aeroelastic characteristics of the structure; an example is given by the transit of trains on a railway bridge, as it determines a change of the dynamic characteristics of a flexible superstructure.

In the framework of PBWE, specific attention has to be given to the choice and probabilistic characterization of the minimum number of “sufficient” and “efficient” [7] parameters that are needed to characterize the wind field and the structural response. In what follows, the parameters that characterize the wind field in the environment are grouped in the *Intensity Measure* vector (**IM**); the parameters that characterize the structural system characteristics are grouped in the vector of *Structural Parameters* (**SP**); the wind-structure interaction is described by the vector of *Interaction Parameters* (**IP**).

In general, the uncertainties are related to: (i) the intrinsic variability of the dynamic characteristics of the structure and the basic parameters of the wind field, arising e.g. from the unpredictable nature of magnitude and direction of the wind velocity and turbulence intensity (*inherent* or *aleatory* uncertainty); (ii) the errors associated to the experimental measures and the incompleteness of data and information (*epistemic* uncertainty); (iii) the modelling of wind actions and their effects on structural response (*model* uncertainty).

In what follows, it is assumed that the two groups of parameters **IM** and **SP** that characterize respectively the wind field in the environment and the structure are not correlated; moreover, the aleatory uncertainty of the structural parameters **SP** might be considered negligible in comparison with the aleatory uncertainty of the environmental parameters **IM**. However, in the *exchange zone* the parameters **IP** describing the wind-structure interaction are strongly dependent on both the environmental **IM** and structural **SP** parameters. Therefore, to derive the probability density functions of the parameters **IP**, the uncertainty of the environmental **IM** and structural **SP** parameters must be taken into account. Assuming then:

$$P(\mathbf{IM} | \mathbf{IP}) = P(\mathbf{IM} | \mathbf{SP}) = P(\mathbf{IM}) \quad (1)$$

$$P(\mathbf{SP} | \mathbf{IP}) = P(\mathbf{SP} | \mathbf{IM}) = P(\mathbf{SP}) \quad (2)$$

where $P(\cdot | \cdot)$ is a conditional probability, the joint probability of **IM**, **IP** and **SP** is given by:

$$P(\mathbf{IM}, \mathbf{IP}, \mathbf{SP}) = P(\mathbf{IP} | \mathbf{IM}, \mathbf{SP}) \cdot P(\mathbf{IM}) \cdot P(\mathbf{SP}) \quad (3)$$

In general, with regard to the uncertainty of the models assumed for the wind actions and the structural behaviour, two elements should be considered: (i) the consistency and predictive capability of the chosen models (model *efficiency*) and (ii) the uncertainty related to the accuracy of the estimate of the relevant model parameters and the response quantities (model *accuracy*). The model efficiency can be improved by varying the selected model class; the model accuracy can be classified as a sort of epistemic uncertainty. It should be noted that it is impossible to obtain the same levels of both efficiency and accuracy for the models representing the wind actions in the environment, the structural behaviour and the wind-structure interaction; moreover, the accuracy of the estimate of the interaction parameters **IP** depends on the accuracy of the estimate of both **IM** and **SP** parameters. In the numerical example developed in the following, the attention is focused on the efficiency of the model of wind-structure interaction.

Examples of the sources of uncertainty that are relevant in Aeolian risk assessment are shown in Table 1. The elements that are taken into account are the location

Table 1 Examples of different sources of uncertainty relevant in Aeolian risk assessment

Location	Typology	Parameter	Propagation
Environment	Aleatory	Wind velocity, direction and turbulence intensity, roughness length, ...	
	Epistemic	Experimental measures of the listed quantities	
	Model	Stochastic models of both wind velocity and direction and turbulent wind (e.g. a Gaussian stationary multi-variate process); wind spectra;	
	Aleatory	Aerodynamic coefficients, Strouhal number, fetch length, ...	
Exchange zone	Epistemic	Experimental measures of the listed quantities	
	Model	Aeroelastic forces, ...	
Structural system	Aleatory	Structural stiffness, strength, mass and damping; geometric quantities; ...	
	Epistemic	Experimental measures of the listed quantities	
	Model	Dynamic model of the structure, model of viscous damping, ...	
= propagation			= no propagation

(environment – exchange zone – structural system), the type of uncertainty (aleatory, epistemic, model), and the correlation among the groups of parameters, i.e. the propagation of uncertainty.

In the framework of PBWE, the aleatory uncertainty of the parameters **IM** can be characterized by defining the joint probability distribution function with reference to a proper time interval (typically, 1 year for high-level performances that are related to serviceability; the whole service life for low-level performances that are associated to ultimate limit states and structural reliability). Elements of **IM** are the mean wind velocity $V_{z_{ref}}$ at a reference height z_{ref} and the direction Θ of the mean velocity. In numerical applications, for serviceability evaluation the aleatory uncertainty of both $V_{z_{ref}}$ and Θ is described by the joint distribution function [8]:

$$\begin{aligned} P_{V_{z_{ref}}, \Theta}(v, \theta) &= P_{\Theta}(\theta) \cdot \left(\exp \left[- \left(\frac{v}{c(\theta)} \right)^{k(\theta)} \right] \right) \\ &= \iint f_{\Theta}(\theta) \cdot f_{V_{z_{ref}}, \Theta}(v, k(\theta), c(\theta)) \cdot dv \cdot d\theta \end{aligned} \quad (4)$$

where

$$f_{V_{z_{ref}}, \Theta}(v, \theta) = \frac{k(\theta)}{c(\theta)} \cdot \left(\frac{v}{c(\theta)} \right)^{k(\theta)-1} \cdot \exp \left[- \left(\frac{v}{c(\theta)} \right)^{k(\theta)} \right] \quad (5)$$

$k(\theta)$ and $c(\theta)$ are the parameters of a Weibull density function and are dependent on Θ . If the reference period is equal to the service life, the distribution of the annual maxima of wind velocity is a Gumbel type I distribution, given by [9]:

$$F_{V_{max}} = P(V_{max} \leq V | \alpha, \beta) = \exp(-\exp[-\alpha \cdot (V - \beta)]) \quad (6)$$

The parameters α and β are equal to

$$\alpha = \frac{\ln 2}{2 \cdot b_1 - E[V_{max}]} \quad (7)$$

$$\beta = E[V_{max}] - \frac{\gamma}{\alpha} \quad (8)$$

$E[V_{max}]$ is the mean of the maximum values ($V_1^{max}, V_2^{max}, \dots, V_n^{max}$) of the wind velocity evaluated over a time interval of 1 year, γ is the Euler constant (equal to 0.577216) and

$$b_1 = \frac{1}{n} \sum_{i=1}^n \frac{i-1}{n-1} V_i^{max} \quad (9)$$

is computed by sorting the maximum values ($V_i^{max}, i = 1, \dots, n$) in ascending order from the minimum ($i = 1$) to the maximum ($i = n$) value.

As concerns the vector **SP**, typical random variables are the structural damping, stiffness, strength and mass, the geometric dimensions and the magnitude of non-environmental loads. The aleatory and epistemic uncertainties can be characterized by joint probability distribution functions, with parameters estimated by experimental data. However, in the following the structural parameters **SP** are considered as deterministic.

The probabilistic characterization of the interaction parameters **IP** may be carried out by defining a conditional probability density function that takes into account the dependence on the parameters **IM** and **SP**. Assuming that an interaction parameter $IP^{(1)}$ can be modeled as a Gaussian random variable, the mean and the standard deviation are given functions of the parameters **IM** and **SP**, and the conditional probability density function is given by Eq. 10:

$$f\left(IP^{(1)} \mid \mathbf{IM}, \mathbf{SP}\right) = \frac{1}{\sqrt{2\pi \cdot \sigma_{IP^{(1)}}^2(\mathbf{IM}, \mathbf{SP})}} \cdot \exp \left[- \left(\frac{IP^{(1)} - \mu_{IP^{(1)}}(\mathbf{IM}, \mathbf{SP})}{\sigma_{IP^{(1)}}(\mathbf{IM}, \mathbf{SP})} \right)^2 \right]. \quad (10)$$

3 A Procedure for Aeolian Risk Assessment

In the PBWE procedure adopted in this paper [10], the structural risk is defined as the probability of exceeding a threshold level of a relevant Decision Variable DV :

$$G(DV) = \iiint \iiint G(DV \mid DM) \cdot f(DM \mid EDP) \cdot f(EDP \mid \mathbf{IM}, \mathbf{IP}, \mathbf{SP}) \cdot f(\mathbf{IP} \mid \mathbf{IM}, \mathbf{SP}) \cdot f(\mathbf{IM}) \cdot f(\mathbf{SP}) \cdot dDM \cdot dEDP \cdot d\mathbf{IM} \cdot d\mathbf{IP} \cdot d\mathbf{SP} \quad (11)$$

where DM is a scalar damage measure and EDP is a scalar engineering demand parameter, representing the structural response (the formal extension to vectors **DM** and **EDP** is straightforward). By means of Eq. 11, the problem of risk assessment is disaggregated into the following elements:

- Site and structure-specific hazard analyses, that is, the assessment of the probability density functions $f(\mathbf{IM})$, $f(\mathbf{SP})$ and $f(\mathbf{IP} \mid \mathbf{IM}, \mathbf{SP})$;
- Structural analysis, aimed at assessing the probability density function of the structural response $f(EDP \mid \mathbf{IM}, \mathbf{IP}, \mathbf{SP})$ conditional on the parameters characterizing the wind field, the wind-structure interaction and the structural properties;
- Damage analysis, that gives the damage probability density function $f(DM \mid EDP)$ conditional on EDP ;
- Finally, loss analysis, that is the assessment of $G(DV \mid DM)$, where $G(\cdot)$ is a complementary cumulative distribution function.

If the performance is expressed by the fulfillment of a limit state, and the limit state condition in terms of an *EDP*, the whole procedure simply requires the evaluation of the probability of exceedance:

$$G(EDP) = \iiint G(EDP | \mathbf{IM}, \mathbf{IP}, \mathbf{SP}) \cdot f(\mathbf{IP} | \mathbf{IM}, \mathbf{SP}) \cdot f(\mathbf{IM}) \cdot f(\mathbf{SP}) \cdot d\mathbf{IM} \cdot d\mathbf{IP} \cdot d\mathbf{SP} \quad (12)$$

In the following, only the mean wind velocity in a fixed direction at a certain height is assumed as *IM* and the parameters *SP* are considered deterministic; Eq. 12 becomes

$$G(EDP) = \iint G(EDP | IM, \mathbf{IP}) \cdot f(\mathbf{IP} | IM) \cdot f(IM) \cdot dIM \cdot d\mathbf{IP} \quad (13)$$

There are several methods for computing the integrals (11–13). In the numerical example below, a crude Monte Carlo simulation is used.

4 The Relevance of Uncertainties with Reference to a Case Example

The relevance of both the epistemic uncertainty of the parameters *IP* (the aerodynamic coefficients in the specific case) and the uncertainty of the models of wind-structure interaction is investigated; moreover, the relevance of model uncertainty on risk assessment is also considered.

The case example makes reference to a bridge design derived from a preliminary design of the suspension bridge over the Messina Strait [11]. The main span of the bridge is 3,300 m long, while the total length including the side spans is 3,666 m. The deck, 60 m wide, is composed by three box sections, the outer ones for the roadways and the central one for the railway. The roadway deck has three lanes for each carriageway (two driving lanes and one emergency lane), each 3.75 m wide, and the railway section has two tracks. The two towers are 383 m high and the bridge suspension system relies on two pairs of steel cables, each with a diameter of 1.24 m and a total length, between the anchor blocks, of approximately 5,000 m. The main characteristics of the structure are summarized in Fig. 2.

The dynamic response and the stability under wind actions of the bridge have been evaluated by a series of analyses carried out in time domain on a finite element model of the bridge: the total number of elements (beams, no compression cable elements and gaps) is 1,614, and the number of nodes is 1,140 [10]. A Newmark time integration scheme has been adopted, to take into account nonlinear and second order effects. The incident turbulent wind velocity time series have been generated as components of a multivariate, multidimensional Gaussian stationary stochastic process [12].

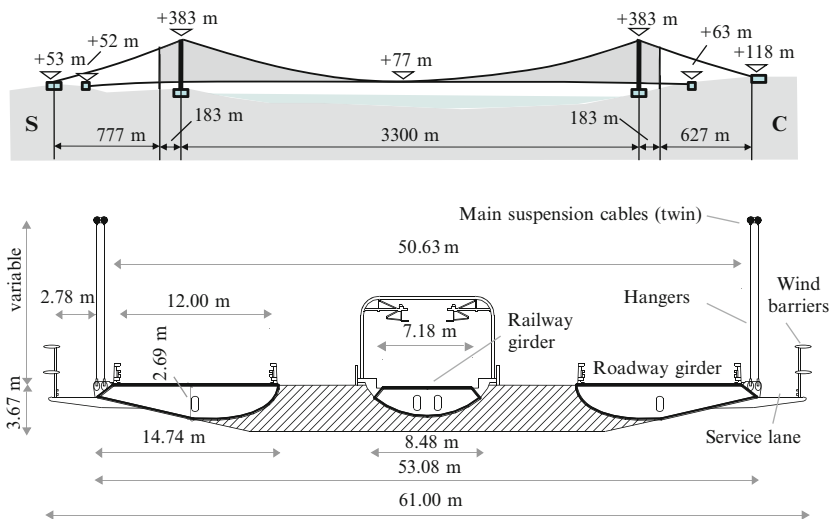


Fig. 2 Diagrammatic longitudinal and transversal sections of the long-span suspension bridge taken as example case (defined according to the considered design derived from a 2005 preliminary design of the Messina Strait Bridge) (C = Calabria; S = Sicily)

4.1 The Epistemic Uncertainty of the Aerodynamic Coefficients

Several types of cross section of the bridge deck, which were obtained by varying the shape and the configuration of the traffic and wind barriers, have been tested in wind tunnels: the multi-box section depicted in Fig. 2 has been chosen in order to optimize the aerodynamic response of the deck [13]. In Fig. 3, two different sets of static aerodynamic coefficients varying with the angle of attack (referred to as “polar lines” in what follows) obtained by experimental tests carried out at different scale on the chosen configuration are reported.

The role of the epistemic uncertainty of the values of the aerodynamic polar lines has been investigated by a parametric analysis of the flutter stability. The analyses have been carried out by using samples of polar lines obtained as a linear combination of the two sets reported in Fig. 3, and identified as polar lines *A* (Fig. 3, left) and polar lines *B* (Fig. 3, right), by:

$$\begin{aligned} c_L(\alpha) &= c_{L,A}(\alpha) \cdot P_L + c_{L,B}(\alpha) \cdot (1 - P_L) \\ c_M(\alpha) &= c_{M,A}(\alpha) \cdot P_M + c_{M,B}(\alpha) \cdot (1 - P_M) \end{aligned} \quad (14)$$

where: $c_{i,j}$ ($i = L, M$ and $j = A, B$) is the aerodynamic coefficient i corresponding to the polar line j ; α is the angle of incidence; P_L and P_M are combination parameters, that vary between 0 and 1.

Critical flutter velocities V_{crit} obtained by the Quasi-Steady formulation of the aeroelastic forces [14] are shown in Fig. 4; the points marked by a cross have been evaluated by FE analyses, the others have been derived by linear interpolation.

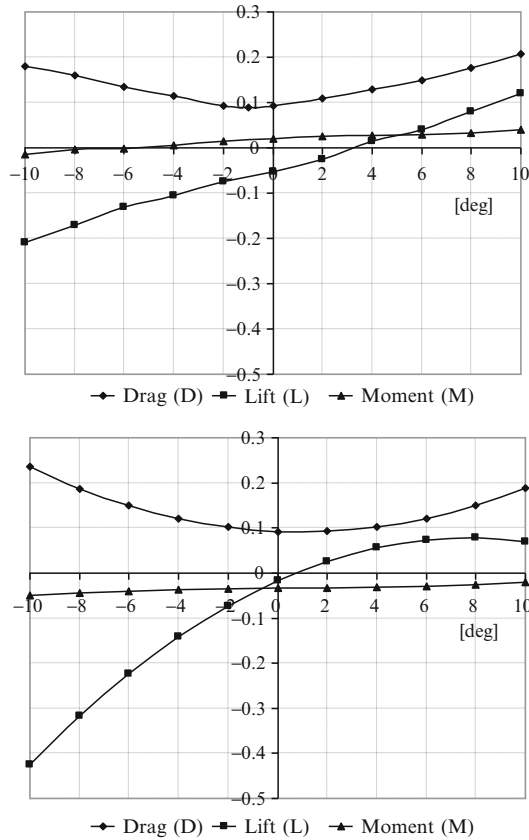


Fig. 3 Examples of polar lines (type A – upper; type B – lower) obtained by experimental tests

The response surface in Fig. 4 shows that the uncertainty of the interaction parameters cannot be disregarded, and that the influence of the uncertainty of the moment coefficient $c_M(\alpha)$ is more relevant than the uncertainty of the lift coefficient $c_L(\alpha)$.

4.2 The Model Uncertainty of Aeroelastic Forces

A general formulation of the equations of motion for a structure subject to aeroelastic effects and modeled as an n degrees of freedom (DOFs) system is given by:

$$\mathbf{M} \cdot \ddot{\mathbf{q}}(t) + \mathbf{C} \cdot \dot{\mathbf{q}}(t) + \mathbf{K} \cdot \mathbf{q}(t) = \mathbf{F}(\text{body shape}; \mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}; \mathbf{V}; t; \boldsymbol{\omega}) \quad (15)$$

where: $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices; $\mathbf{q}$, $\dot{\mathbf{q}}$, $\ddot{\mathbf{q}}$ are the vectors of system DOFs and their first and second time derivatives; $\mathbf{V}$ is the wind velocity vector; $\boldsymbol{\omega}$ is the vector of the natural frequencies of the system. The components of the forcing functions $\mathbf{F}$ that depend on the body motion ($\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}; t; \boldsymbol{\omega}$) are the “self-excited” or “aeroelastic” components $\mathbf{F}_{se}$.

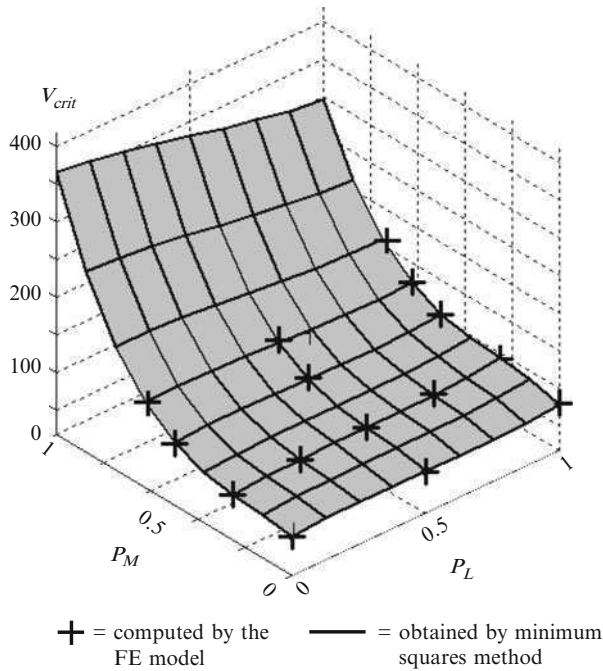


Fig. 4 Response surface of the critical flutter wind velocity

Both frequency and time domain techniques have been implemented to model the aeroelastic forces [15]; time domain approaches allow taking directly into account the nonlinear response of the structure. The analysis consists of a time integration that involves a step-by-step updating of the kinematic parameters and acting forces. Referring to Fig. 5, where the problem is represented as two-dimensional, the horizontal and vertical components of the wind velocity V are considered as composed by the mean components U and W and the fluctuating turbulent components $u(t)$ and $w(t)$. The velocity is not horizontal, and has a time-varying angle of incidence α . Adopting the notation introduced in Eq. 15 and Fig. 5, and assuming that:

$$\mathbf{q}^T = [p \ h \ \vartheta]$$

the aeroelastic forces $\mathbf{F}_{se}$ can be expressed as:

$$\mathbf{F}_{se}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}; t; \omega) = \mathbf{P}(t, \omega) \cdot \ddot{\mathbf{q}} + \mathbf{Q}(t, \omega) \cdot \dot{\mathbf{q}} + \mathbf{R}(t, \omega) \cdot \mathbf{q} \quad (16)$$

where $\mathbf{P}(t, \omega)$, $\mathbf{Q}(t, \omega)$ and $\mathbf{R}(t, \omega)$ are matrices of coefficients that depend on t and ω .

The choice of the model of the aeroelastic forces is not trivial: great difficulties can be found in the estimation of the aerodynamic delay, that is, the transient effect due to the adjustment of the aerodynamic field as a consequence of a change in the

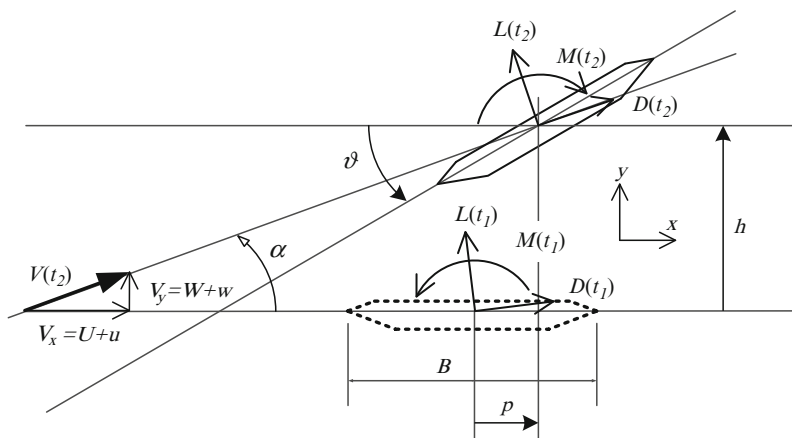


Fig. 5 Typical bridge deck section under turbulent wind actions

geometric configuration (due to the rotation or displacement of the deck). Therefore different formulations, characterized by an increasing level of complexity, have been implemented, as illustrated in [14].

According to non-aeroelastic (NO) theory, aeroelastic effects are disregarded and the angle of incidence changes with time only due to the turbulence of the incident wind; the components of $\mathbf{F}$ are the drag, lift and moment terms, expressed in terms of the coefficients:

$$\begin{aligned} D(t) &= \frac{1}{2} \rho \cdot |V(t)|^2 \cdot B \cdot c_D[\alpha(t)] \\ L(t) &= \frac{1}{2} \rho \cdot |V(t)|^2 \cdot B \cdot K_{L0} \cdot \alpha(t) \\ M(t) &= \frac{1}{2} \rho \cdot |V(t)|^2 \cdot B^2 K_{M0} \cdot \alpha(t) \end{aligned} \quad (17)$$

where: ρ is the air mass density; $V(t)$ is the instantaneous velocity; $\alpha(t)$ is the instantaneous inclination of the wind velocity vector on the horizontal axis; B is a characteristic dimension of the cross section (Fig. 5); c_D is the drag coefficient; K_{L0} and K_{M0} are the angular coefficients of the lift and moment polar lines (evaluated at $\alpha(t) = 0$).

According to the steady (ST) theory, the angle of incidence changes with time due to both the incident wind turbulence and the torsional rotation of the deck. Assuming that (i) the bridge deck section rotates around a mean equilibrium position ($\vartheta = \vartheta_0$) and (ii) the displacements are small (that is, both lift and moment polar lines can be linearised), the aerodynamic coefficients are given by

$$\begin{aligned} c_L(\gamma) &= c_L(\vartheta_0) + K_{L0} \cdot (\gamma - \vartheta_0) \\ c_M(\gamma) &= c_M(\vartheta_0) + K_{M0} \cdot (\gamma - \vartheta_0) \end{aligned} \quad (18)$$

where $\gamma(t) = \alpha(t) - \vartheta(t)$ and K_{L0}, K_{M0} are the angular coefficients of the polar lines evaluated at $(\vartheta = \vartheta_0)$. The components of $\mathbf{F}$ are expressed in terms of

$$\begin{aligned} D(t) &= \frac{1}{2} \rho \cdot |V(t)|^2 \cdot B \cdot c_D[\gamma(t)] \\ L(t) &= \frac{1}{2} \rho \cdot |V(t)|^2 \cdot B \cdot c_L[\gamma(t)] \\ M(t) &= \frac{1}{2} \rho \cdot |V(t)|^2 \cdot B^2 \cdot c_M[\gamma(t)] \end{aligned} \quad (19)$$

Adopting the general formulation of Eq. 16, one obtains:

$$\mathbf{F}_{se} = \mathbf{F}_{se}(\mathbf{q}; t) = \mathbf{R} \cdot \mathbf{q}(t) \quad (20)$$

The steady theory has the advantage of simplicity; furthermore, in case of non turbulent incident wind, it interprets the fundamental mechanism of a flutter stability problem (coalescence of frequencies, influence of structural parameters, etc.). Nevertheless it implies many approximations, related to both neglecting the dependence of aeroelastic forces on structural velocities, accelerations and oscillation frequencies and the linearization of the relation between aeroelastic forces and structural DOFs. Furthermore, the steady theory does not consider the aerodynamic delay, and utilizes static aerodynamic coefficients.

According to the quasi-steady (QS) theory, the instantaneous aeroelastic forces acting on the structure are the same that act on the structure when it moves with constant translational and rotational velocities, equal to the actual instantaneous ones. The main assumption at the base of the theory consists in considering that the body (the deck section) is quiescent, and that the wind has a velocity and acts in a direction coincident with the instantaneous relative (wind-deck) value: such assumption is represented in Fig. 6. Adopting the hypothesis of small displacements around the

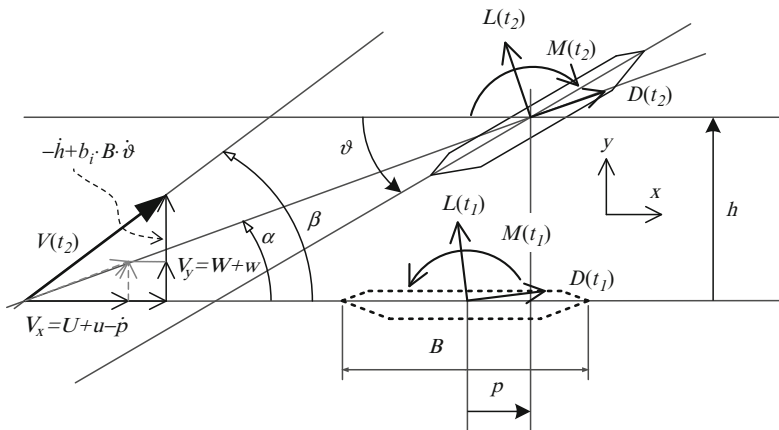


Fig. 6 Quasi-steady theory assumption

mean configuration and the notation of Fig. 6, Eqs. 18 and 19 are still valid and the expressions of the aeroelastic forces are identical to those derived by steady theory, with

$$\beta_i(t) = \arctg \left(\frac{V_y(t) - \dot{h} + b_i \cdot B \cdot \dot{\vartheta}(t)}{V_x - \dot{p}} \right) \quad (i = L, M) \quad (21)$$

in place of $\alpha(t)$. Therefore:

$$\begin{aligned} D(t) &= \frac{1}{2} \rho \cdot |V_L(t)|^2 \cdot B \cdot c_D [\gamma(t)] \\ L(t) &= \frac{1}{2} \rho \cdot |V_L(t)|^2 \cdot B \cdot c_L [\gamma_L(t)] \\ M(t) &= \frac{1}{2} \rho \cdot |V_M(t)|^2 \cdot B^2 \cdot c_M [\gamma_M(t)] \end{aligned} \quad (22)$$

where the terms V_i (for: $i = L, M$) represent the relative velocities and

$$\begin{aligned} \gamma_i(t) &= \beta_i(t) - \vartheta(t) \\ |V_i(t)|^2 &= \sqrt{(V_x - \dot{p})^2 + \left(V_y - \dot{h} + b_i \cdot B \cdot \dot{\vartheta}(t) \right)^2} \quad (i = L, M) \end{aligned}$$

The coefficients b_i ($i = L, M$) are derived by wind tunnel tests [16] or by Computational Fluid Dynamics [17, 18]. Adopting the general formulation of Eq. 15, one obtains:

$$\mathbf{F}_{se} = \mathbf{F}_{se}(\mathbf{q}, \dot{\mathbf{q}}; t) = \mathbf{R} \cdot \mathbf{q}(t) + \mathbf{Q} \cdot \dot{\mathbf{q}}(t) \quad (23)$$

The quasi-steady theory considers the dependence of the aeroelastic forces on the structural velocities, and has a relatively simple algorithmic implementation. The dependence on oscillation frequencies is neglected, and the relation between aeroelastic forces and the DOFs of the structure is linear. The quasi-steady theory does not consider the aerodynamic delay, and utilizes static aerodynamic coefficients, with the possible exception for the b_i coefficients ($i = L, M$), whose value can be assessed by dynamic tests [19].

In the modified quasi-steady theory (QSM), the aerodynamic lift and moment coefficients are considered variable with time and measured by wind tunnel tests [20]. Referring to Fig. 6, aeroelastic forces are still expressed by Eq. 22 but the aerodynamic coefficients c_L and c_M are given by:

$$\begin{aligned} c_L &= c_L(\vartheta_0) + \int_{\vartheta_0}^{\vartheta} K_L d\bar{\vartheta} \\ c_M &= c_M(\vartheta_0) + \int_{\vartheta_0}^{\vartheta} K_M d\bar{\vartheta} \end{aligned} \quad (24)$$

where $c_L(\vartheta_0)$ and $c_M(\vartheta_0)$ are the static aerodynamic coefficients referred to the mean equilibrium configuration ($\vartheta = \vartheta_0$), and K_L , K_M are the so-called dynamic derivatives:

$$K_L = h_3 \cdot \left(\frac{\partial c_L}{\partial \vartheta} \right)_{\vartheta=\vartheta_0} \quad K_M = a_3 \cdot \left(\frac{\partial c_M}{\partial \vartheta} \right)_{\vartheta=\vartheta_0} \quad (25)$$

h_3 and a_3 are coefficients [21] assessed by dynamic wind tunnel tests and dependent on both the angle of rotation of the deck and a reduced wind velocity

$$V_{red} = \frac{V}{\omega_f \cdot B}$$

that is a function of the circular frequency of motion ω_f .

For multi-degree of freedom structures, the motion frequency ω_f is a combination of the natural frequencies that varies with time, depending on the state of structure, if the behavior is nonlinear. Therefore in advance computation of h_3 and a_3 is not possible; to overcome this problem, the fundamental frequency of the structure is used to compute the reduced velocity and the coefficients h_3 and a_3 . Moreover, the dependence of aeroelastic forces on the motion frequency is not considered. According to the general formulation, one can write

$$\mathbf{F}_{se} = \mathbf{F}_{se}(\mathbf{q}, \dot{\mathbf{q}}; t) = \mathbf{R}(t) \cdot \mathbf{q}(t) + \mathbf{Q}(t) \cdot \dot{\mathbf{q}}(t) \quad (26)$$

The QSM theory has the attractive aspects of the QS theory but implements dynamic aerodynamic coefficients. Such coefficients take into account the nonlinearity of the response and roughly the aerodynamic delay.

To examine the relevance of the effects of the model uncertainty, in Figs. 7 and 8 samples of the time-histories of the transversal (along wind) and vertical (across wind) displacements U_y and U_z of the centre of mass of the cross section at bridge midspan are shown; they are evaluated by considering the four models (NO – ST – QS – QSM), for an incident turbulent flow with a mean velocity $V_m(z_{deck})$ at the height of the deck at midspan ($z_{deck} = 77$ m) equal to 45 m/s.

The histograms of the magnitude of the displacements and the corresponding probability density functions are also reported in the same figures. It can be noted that by increasing the model complexity (from NO to QSM), both the mean values and the variances of the displacement decrease.

The analyses have been repeated for mean incident wind velocities equal to 21 and 57 m/s. In Fig. 9, the distributions along the whole length of the bridge of the maximum values of the transversal and vertical displacements are reported for all three values of the velocity (note that the values are not simultaneous). The plots confirm the trend just highlighted: by increasing the complexity of the aeroelastic force models, the envelopes flatten. Moreover, ST is the most sensitive model to an increase of the mean wind velocity.

The analyses show that the uncertainty that characterize the model of the aeroelastic forces has a great relevance on the maximum values of the response

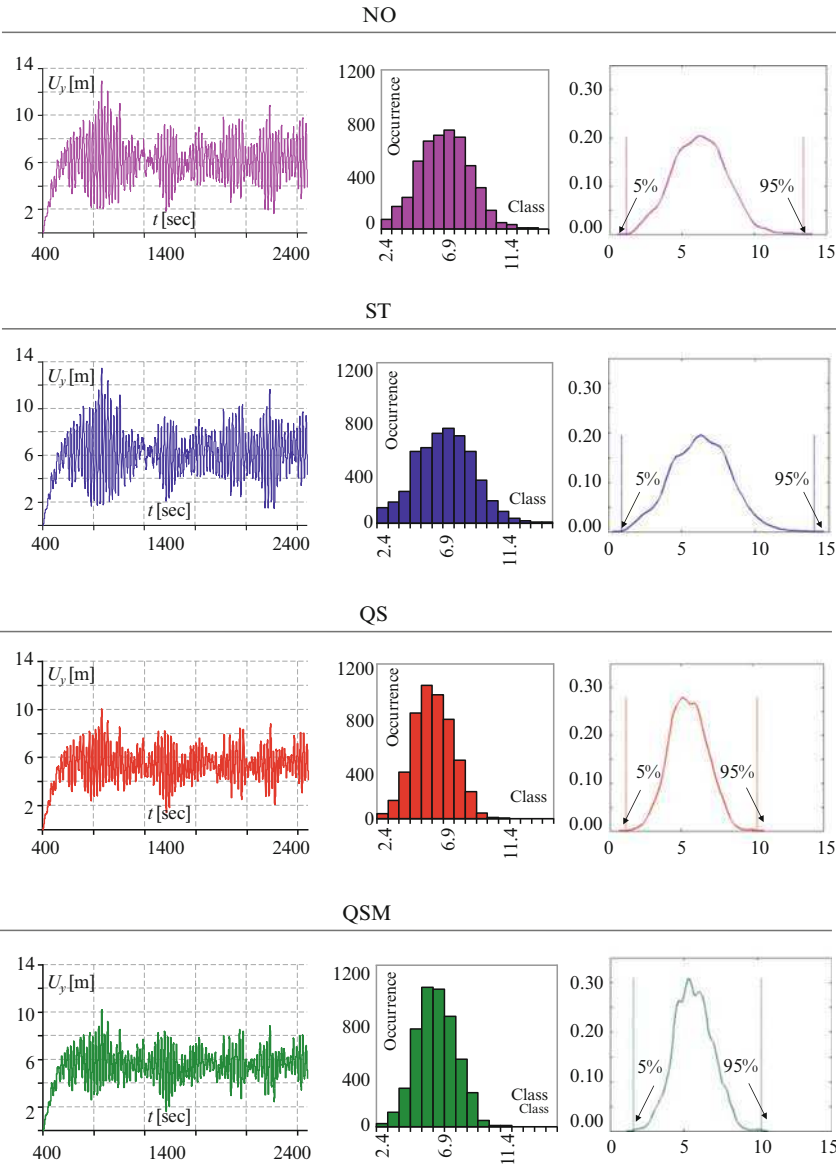


Fig. 7 Transverse displacement U_y of the centre of mass of the bridge cross section at midspan evaluated for $V_m(z_{deck}) = 45 \text{ m/s}$: for each model (NO – ST – QS – QSM) of the aeroelastic forces, sample of the time history (left), histogram of instantaneous values (centre) and probability density function (right)

parameters: for example, the difference between the vertical displacements derived by ST and QSM theories is even equal to 80% (in case of a mean velocity equal to 57 m/s).

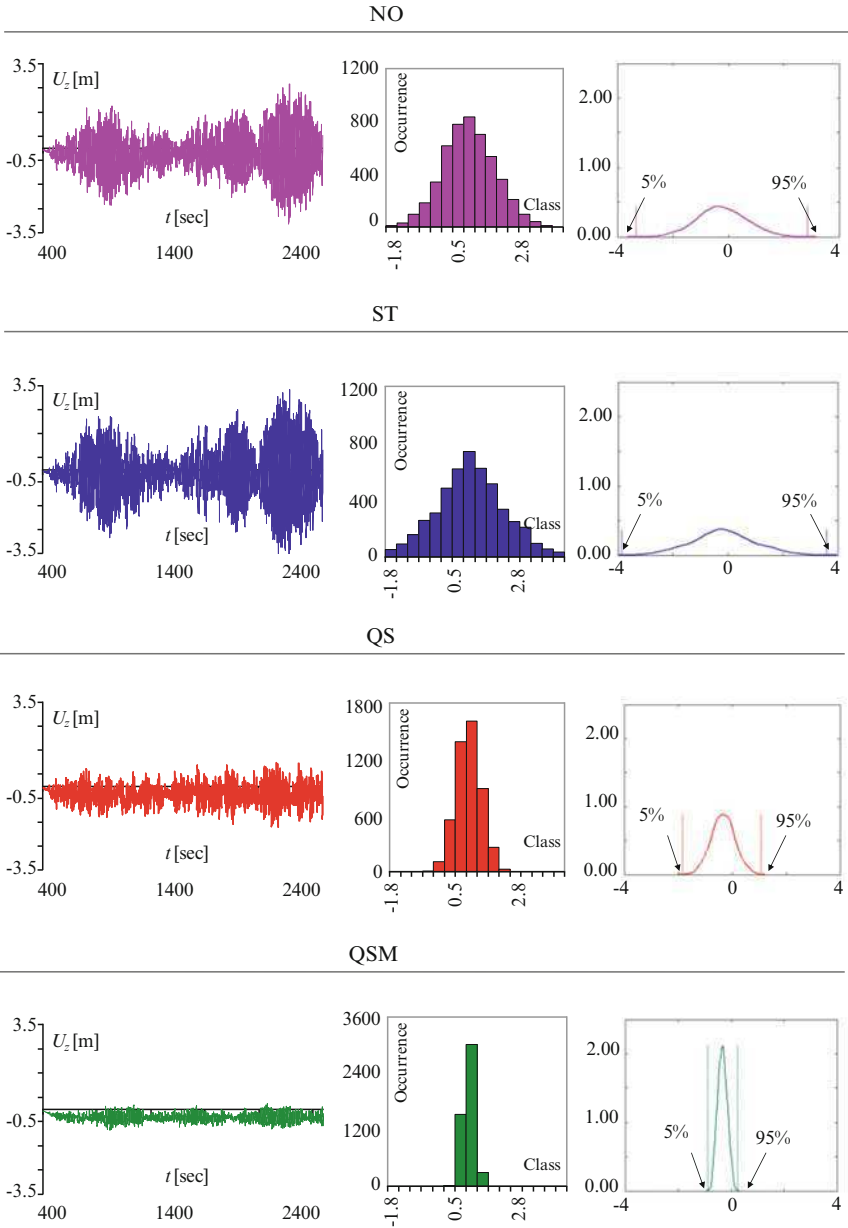


Fig. 8 Vertical displacement U_z of the centre of mass of the bridge cross section at midspan evaluated for $V_m(z_{deck}) = 45 \text{ m/s}$: for each model (NO – ST – QS – QSM) of the aeroelastic forces, sample of the time history (*left*), histogram of instantaneous values (*centre*) and probability density function (*right*)

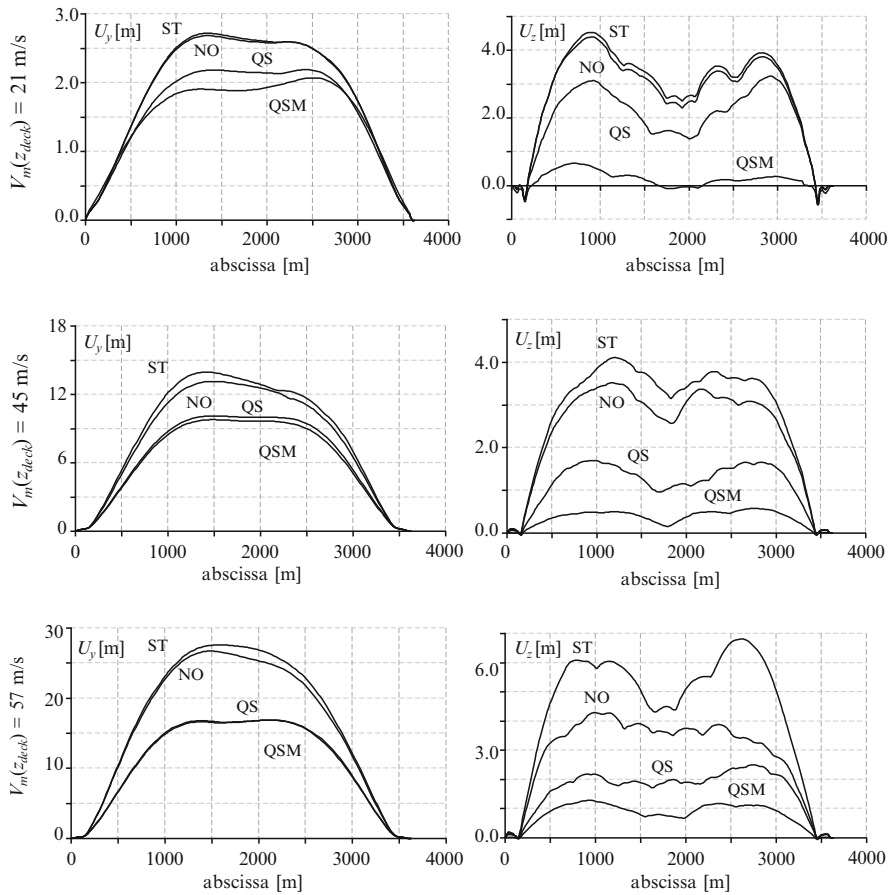


Fig. 9 Envelopes of deck displacements ($V_m(z_{deck}) = 21 - 45 - 57$ m/s): transversal maximum displacements (left) and vertical maximum displacements (right)

4.3 The Influence of Model Uncertainty on Risk Assessment

To evaluate the influence of model uncertainty on risk assessment, assume that (a) the interaction **IP** and structural **SP** parameters are deterministic and (b) the only **IM** is the mean wind velocity evaluated at the deck height $V_m(z_{deck})$. With reference to Eq. 12, a formulation of the complementary cumulative distribution function (CCDF) of **EDP** is simply given by:

$$\begin{aligned}
 G(EDP) &= 1 - F(EDP) = 1 - \int f(EDP) \cdot dEDP \\
 &= 1 - \int \left[\int f(EDP|IM) \cdot f(IM) \cdot dIM \right] \cdot dEDP \quad (28)
 \end{aligned}$$

The function is evaluated by Monte Carlo simulation (considering 5,000 samples). For each run, the value of $V_m(z_{deck})$ is obtained (a) by sampling a value of $V_m(z = 10 \text{ m})$ by a Weibull CDF (reference period = 1 year):

$$F(V_m(z = 10 \text{ m})) = 1 - \exp \left[-\frac{1}{2} \left(\frac{V_m(z = 10 \text{ m})}{\sigma} \right)^k \right] \quad (29)$$

with parameters σ and k set respectively equal to 6.02 m/s and 2.02; (b) by assuming an homogeneous roughness length z_0 (equal to 0.5 m), and (c) by describing the variation of $V_m(z)$ with the height z by a logarithmic law [22, 23].

The response parameters EDP are:

- the mean $U_y \text{ mean}$ and the maximum $U_y \text{ max}$ values of the transversal displacements of the centre of mass of the cross section of the deck at midspan;
- the mean $U_z \text{ mean}$, the maximum $U_z \text{ max}$ and the minimum $U_z \text{ min}$ values of the vertical displacements of the same point.

In evaluating the function $f(EDP|IM)$, the effects of the model uncertainty associated to the definition of the aeroelastic forces F_{se} are taken into account by characterizing the distribution parameters (e.g. mean and variance) on the basis of the results of the numerical analyses illustrated in the previous Section. It is assumed that each EDP is characterized by a Gaussian probability density function:

$$f(EDP|IM) = \frac{1}{\sqrt{2\pi} \cdot \sigma_{EDP}^2(IM)} \cdot \exp \left[-\left(\frac{EDP - \mu_{EDP}(IM)}{\sigma_{EDP}^2(IM)} \right)^2 \right] \quad (30)$$

The laws of variation of the distribution parameters $\mu_{EDP}(IM)$ and $\sigma_{EDP}^2(IM)$ are evaluated by a proper fitting of the mean values and variances obtained for the three values of the mean wind velocity (21, 45 and 57 m/s) and the four aeroelastic models. The values of the parameters and the laws of variation are reported in Table 2:

Table 2 Values of the distribution parameters in Eq. 30 and laws of variation as a function of IM

EDP	$IM = V_m(z_{deck})$	Values of the distribution parameters (μ_{EDP} , σ_{EDP}^2)			Law of variation of the distribution parameters
		21 m/s	45 m/s	57 m/s	
$U_y \text{ mean}$	μ_{EDP}	1.367	5.724	9.380	$0.0028 \cdot IM^2 + 0.0019 \cdot IM$
	σ_{EDP}^2	0.015	0.161	0.441	$6E-6 \cdot IM^3 - 0.0003 \cdot IM^2 + 0.0043 \cdot IM$
$U_y \text{ max}$	μ_{EDP}	2.226	11.659	21.695	$1E-4 \cdot IM^3 - 0.0005 \cdot IM^2 + 0.0705 \cdot IM$
	σ_{EDP}^2	0.279	3.217	35.194	$0.2790 + 1E-14 \cdot IM^{8.85}$
$U_z \text{ mean}$	μ_{EDP}	-0.123	-0.262	-0.301	$2E-5 \cdot IM^2 - 0.0067 \cdot IM$
	σ_{EDP}^2	0.001	0.006	0.010	$3E-5 \cdot IM^2 + 0.0002 \cdot IM$
$U_z \text{ max}$	μ_{EDP}	0.161	1.804	2.681	$0.001 \cdot IM^2 - 0.0078 \cdot IM$
	σ_{EDP}^2	0.012	2.071	2.766	$0.001 \cdot IM^2 - 0.0109 \cdot IM$
$U_z \text{ min}$	μ_{EDP}	-0.414	-2.329	-3.184	$9.0E-4 \cdot IM^2 - 0.0083 \cdot IM$
	σ_{EDP}^2	0.016	1.589	2.121	$8.0E-4 \cdot IM^2 - 0.008 \cdot IM$

it is evident that the variance, which is assumed as a measure of the model uncertainty, increases with the considered IM .

For each sampled value of IM , the parameters of the density function of each EDP are estimated by the relations in Table 2. The corresponding probability density functions are plotted in Figs. 10 and 11; the mean values and the standard deviations of the $EDPs$ increase if V_m increases.

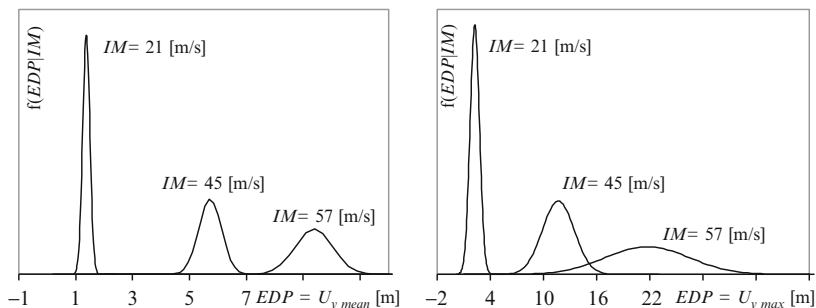


Fig. 10 Probability density functions of the considered $EDPs$ in transversal direction: mean $U_{y\ mean}$ and maximum $U_{y\ max}$ values of the transversal displacements of the centre of mass of the cross section of the deck at midspan evaluated for $V_m(z_{deck}) = 21 - 45 - 57\text{ m/s}$

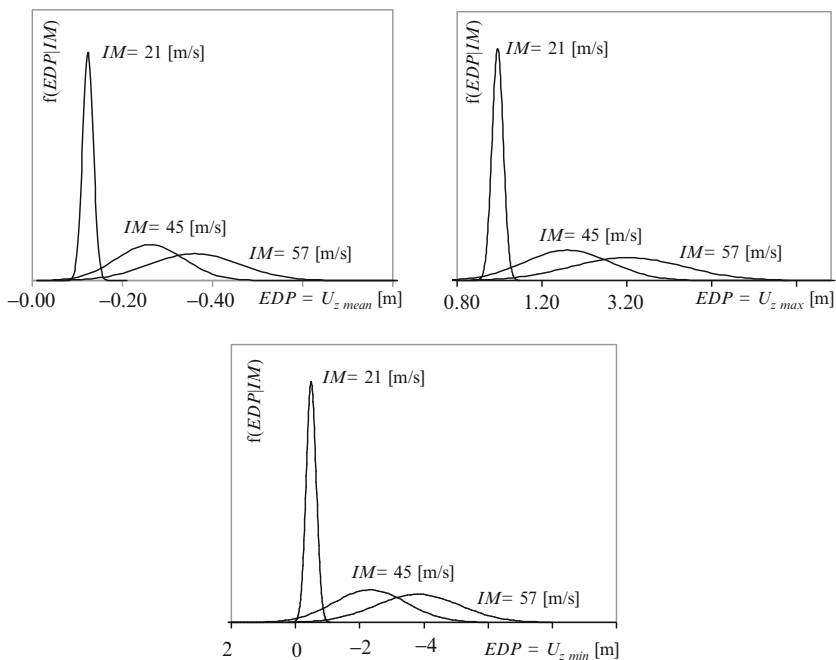


Fig. 11 Probability density functions of the considered $EDPs$ in vertical direction: mean $U_{z\ mean}$, maximum $U_{z\ max}$ and minimum $U_{z\ min}$ values of the vertical displacements of the same point, evaluated for $V_m(z_{deck}) = 21 - 45 - 57\text{ m/s}$

Finally, the CCDFs of the *EDPs* obtained by two different sets of analyses are compared in Figs. 12 and 13: in set 1, a deterministic value (the mean value in Table 2) is attributed to each *EDP*; in set 2, a value of each *EDP* is sampled from the appropriate density function (assuming the same pseudo-random number for all *EDPs* in each run). It is assumed that the analyses in set 2 take into account the

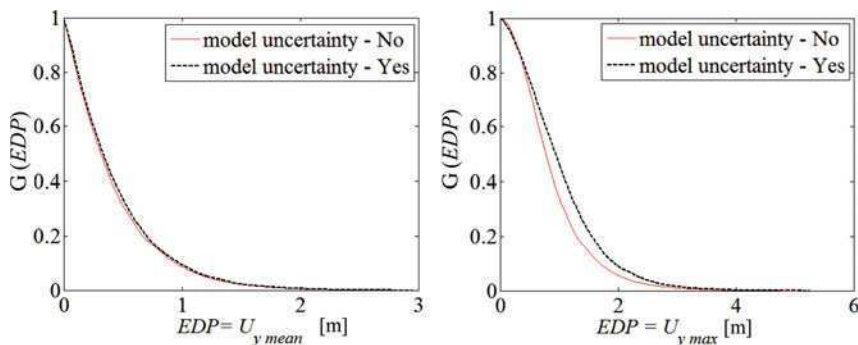


Fig. 12 Complementary cumulative distribution functions of the considered *EDPs* in transversal direction: mean $U_{y \text{ mean}}$ and maximum $U_{y \text{ max}}$ values of the transversal displacements of the centre of mass of the cross section of the deck at midspan, evaluated disregarding or considering the uncertainty of the models of the aeroelastic forces

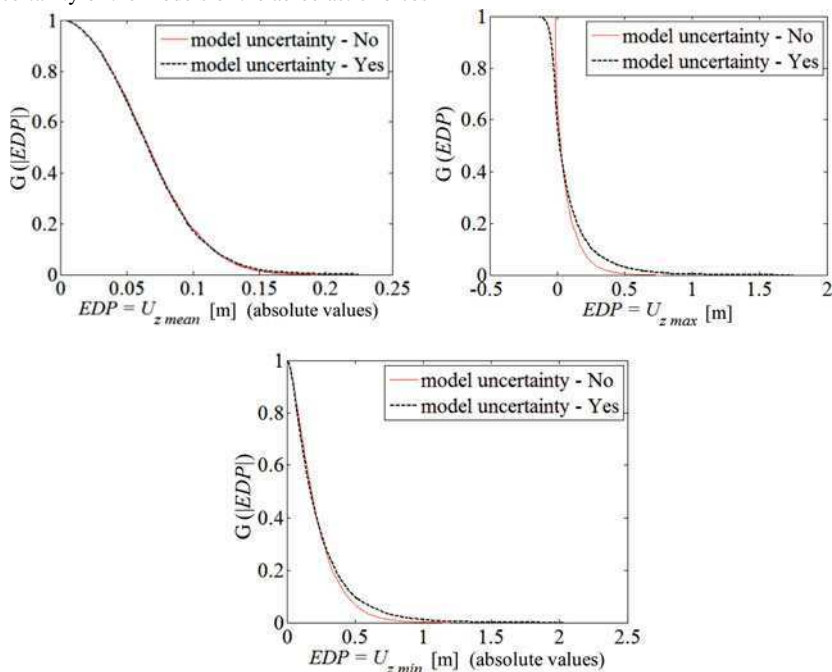


Fig. 13 Complementary cumulative distribution functions of the considered *EDPs* in vertical direction: mean $U_{z \text{ mean}}$, maximum $U_{z \text{ max}}$ and minimum $U_{z \text{ min}}$ values of the vertical displacements of the same point, evaluated disregarding or considering the uncertainty of the models of the aeroelastic forces

effects of model uncertainty on risk assessment, at least in a very rough manner. Looking at the two sets of plots, it results that the model uncertainty can be disregarded if mean displacements are considered, while it has to be taken into account if response parameters related to the turbulence of the wind field are considered.

5 Concluding Remarks

In this paper the problem of the Aeolian risk assessment of slender structures is tackled in the framework of Performance-Based Wind Engineering (PBWE). A classification of the main sources of uncertainty to be dealt with in order to take into account the stochastic nature of wind actions, structural response and their interaction is illustrated in detail; then, the relevance of some uncertainties is estimated with reference to an example case, a long span suspension bridge. The final aim is the improvement of a methodology for Performance-Based Wind Engineering currently under development [10].

Particular attention has been paid on the minimum information that is needed to characterize satisfactorily the wind field and the wind-structure interaction: this is essential, since the data actually available (and likely to be available in the future) are rather limited.

The influence of model uncertainty is investigated with reference to buffeting, by implementing in time domain different formulations of the aeroelastic forces. The relevance of the epistemic uncertainty is investigated in evaluating the flutter instability, by a parametric analysis that assumes different values of the aerodynamic coefficients. Finally, numerical analyses have been carried out to estimate the influence of model uncertainty on risk assessment.

While it appears premature to draw general conclusions, it already appears that both model and epistemic uncertainties affecting the characterization of wind-structure interaction mechanisms must be taken into account in the assessment of Aeolian risk.

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References

1. Augusti, G., Borri, C., Neumann, H.-J.: Is Aeolian risk as significant as other environmental risks? *Reliab. Eng. Syst. Saf.* **74**, 227–237 (2001)
2. Kareem, A.: Wind effects on structures: a probabilistic viewpoint. *Probab. Eng. Mech.* **4**(2), 166–200 (1987)
3. Simiu, E., Scanlan, R.H.: *Wind Effects on Structures: Fundamentals and Applications to Design*, 3rd edn. Wiley-Interscience, New York (1996)
4. Augusti, G., Ciampoli, M.: Performance-based design in risk assessment and reduction. *Probab. Eng. Mech.* **23**, 496–508 (2008)

5. Petrini, F., Ciampoli, M., Augusti, G.: Performance-based wind engineering: risk assessment of a long span suspension bridge. In: Esteva L. (ed.) *Proceedings of the Workshop on Reliability and Optimization of Structural Systems IFIP WG 7.5*, Toluca, Mexico, 6–9 Aug 2008
6. Holmes, J.D.: *Wind Loading of Structures*, 2nd edn. Taylor and Francis, London (2007)
7. Luco, N., Cornell, C.A.: Structure-specific scalar intensity measures for near-source and ordinary earthquake ground motions. *Earthquake Spectra* **23**(2), 357–392 (2007)
8. Xu, Y.L., Liu, T.T., Zhang, W.S.: Buffeting-induced fatigue damage assessment of a long suspension bridge. *Int. J. Fatigue* **31**, 575–586 (2009)
9. Mann J., Kristensen L., Jensen N.O.: Uncertainties of extreme winds, spectra and coherence. In: *Proceedings of the International Symposium on Advances in Bridge Aerodynamics*, Copenhagen, Denmark, 10–13 May 1998
10. Ciampoli, M., Petrini, F., Augusti, G.: Performance-based wind engineering: towards a general procedure, Submitted to *Structural Safety* (2010)
11. Bontempi, F.: Basis of design and expected performances for the Messina Strait Bridge. In: *Proceedings of the International Conference on Bridge Engineering – Challenges in the 21st Century*, Hong Kong, China, 1–3 Nov 2006
12. Carassale, L., Solari, G.: Monte Carlo simulation of wind velocity field on complex structures. *J. Wind Eng. Ind. Aerodyn.* **94**, 323–339 (2006)
13. Pecora, M., Lecce, L., Marulo, F., Corio, D.P.: Aeroelastic behaviour of long span bridges with “multibox” type deck sections. *J. Wind Eng. Ind. Aerodyn.* **18**, 343–358 (1993)
14. Petrini, F., Giuliano, F., Bontempi, F.: Comparison of time domain techniques for the evaluation of the response and the stability in long span suspension bridges. *Comput. Struct.* **85**, 1032–1048 (2007)
15. Caracoglia, L., Jones, N.P.: Time domain vs. frequency domain characterization of aeroelastic forces for bridge deck section. *J. Wind Eng. Ind. Aerodyn.* **91**, 371–402 (2003)
16. Lazzari, M., Vitalini, R.V., Saetta, A.V.: Aeroelastic forces and dynamic response of long-span bridges. *Int. J. Numer. Methods Eng.* **60**, 1011–1048 (2004)
17. Bruno, L., Khris, S.: The validity of 2D numerical simulations of vortical structures around a bridge deck. *Math. Comput. Modell.* **37**, 795–828 (2003)
18. Petrini F., Giuliano F., Bontempi F.: Modeling and simulations of aerodynamic in a long span suspension bridge. In: Augusti G., Schueller G.I., Ciampoli M. (eds.) *Proceedings of the 9th International Conference on Structural Safety and Reliability ICOSSAR’05*, Rome, Italy, 19–23 June 2005
19. Diana, G., Falco, M., Bruni, S., Cigada, A., Collina, A.: Risposta di un ponte a grande luce al vento turbolento: confronto tra simulazioni numeriche e risultati sperimentali (in Italian). In: *Proceedings of the 3rd IN-VENTO Conference*, Rome, Italy, 19–21 Oct (1994)
20. Diana, G., Falco, M., Bruni, S., Cigada, A., Larose, G.L., Damsgaard, A., Collina, A.: Comparison between wind tunnel tests on a full aeroelastic model of the proposed bridge over Stretto di Messina and numerical results. *J. Wind Eng. Ind. Aerodyn.* **54/55**, 101–113 (1995)
21. Zasso, A.: Flutter derivatives: Advantages of a new representation convention. *J. Wind Eng. Ind. Aerodyn.* **60**, 35–47 (1996)
22. Lungu, D., Rackwitz, R.: Joint Committee on Structural Safety – Probabilistic Model Code, Part 2: Loads. <http://www.jcss.ethz.ch/> (2001). Accessed 2 Apr 2010
23. Solari, G., Piccardo, G.: Probabilistic 3-D turbulence modelling for gust buffeting of structures. *Probab. Eng. Mech.* **16**, 73–86 (2001)

The Method of Separation: A Novel Approach for Accurate Estimation of Evolutionary Power Spectra

Dominik Schillinger and Vissarion Papadopoulos

Abstract One of the most widely used techniques for the simulation of Gaussian evolutionary random fields is the spectral representation method. Its key quantity is the power spectrum, which characterizes the random field in terms of frequency content and spatial evolution in a mean square sense. For the simulation of a random physical phenomenon, the power spectrum can be directly obtained from corresponding measured samples by means of estimation techniques. The present contribution starts with a short review of established power spectrum estimation techniques, which are based on the short-time Fourier, the harmonic wavelet and the Wigner–Ville transforms, and subsequently introduces a method for the estimation of separable random fields, called the method of separation. The characteristic drawbacks of the established methods, i.e. the limitation of simultaneous space–frequency localization or the appearance of negative spectral density, lead to poor estimation results, if the Fourier transform of the input samples consists of a narrow band of frequencies. The proposed method of separation, combining accurate spectrum resolution in space with an optimum localization in frequency, considerably improves the estimation accuracy in the presence of strong narrow-bandedness, which is illustrated by a practical example from stochastic imperfection modeling in structures.

1 Introduction

Within the last 3 decades, computational stochastic mechanics has evolved into a self-contained and prolific field of research, which has brought forth a wide range of sophisticated and well-established methodologies for the stochastic simulation of

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uncertain engineering systems (see [4,24] and the references therein). One emerging field of application is the stability analysis of thin-walled structures, where the random variability of geometric and material imperfections leads to considerable uncertainty in corresponding buckling loads [12,13,18,20]. Apart from methodological maturity, the quality of practical stochastic simulations predominantly depends on the accurate reproduction of the random physical key phenomena by corresponding random field models. The most widely used technique for the simulation of imperfections as random fields is the spectral representation method [7,21,23]. The key quantity of spectral representation is the power spectrum [14,16,17], which is related to the average energy of the random field and is obtained in practical applications by estimation from a series of measurements [3,7,13,22]. Despite its decisive importance for realistic stochastic buckling simulations, only little experience exists so far in transferring experimental imperfection measurements, which typically are strongly narrow-band functions at very low frequencies, into accurate evolutionary power spectra.

Against this background, the present contribution intends to shed some light on key issues related to the evolutionary power spectrum estimation of strongly narrow-band random fields, with special emphasis on their application to stochastic imperfection modeling in structures. First, a concise review of the most important existing methods for evolutionary power spectrum estimation is presented, which are based on the short-time Fourier, the harmonic wavelet and the Wigner–Ville transforms. Second, a simple method for evolutionary spectrum estimation of separable random fields is introduced, referred to as the method of separation in the following [19]. Third, the presented methods are applied to the estimation of two benchmark spectra, i.e. the modulated Kanai–Tajimi spectrum [8] and the power spectrum of geometric imperfections in an I-section flange [6]. The results demonstrate the considerable advantage of the method of separation in the presence of strong narrow-bandedness due to its accurate simultaneous space–frequency localization.

2 Relevant Elements of Stochastic Process Theory

A Gaussian random field $h(x)$, equivalently known as a stochastic process in a time context, represents an ensemble of spatial functions, whose exact values are a-priori indeterminate, but follow a Gaussian probability distribution. It can be split into a deterministic mean $\mu(x)$ and a zero-mean Gaussian random field $f(x)$ in the form $h(x) = \mu(x) + f(x)$. In view of the spectral decomposition of its realizations into trigonometric functions, the Gaussian zero-mean random field $f(x)$ can be characterized by a two-sided power spectrum S , which specifies the stochastic frequency content and its correlation along the spatial axis [11,14,16,17,25]. It is called homogeneous, if $S(\omega)$ depends only on frequency ω , and evolutionary, if $S(\omega, x)$ depends on both frequency ω and space x . An intuitive approach to the power spectrum is

its interpretation as the distribution of the mean square of the random field over the space–frequency domain, so that it follows

$$E \left[|f(x)|^2 \right] = 2 \int_0^\infty S(\omega, x) d\omega \quad (1)$$

Analogous to Eq. 1, the incremental energy in frequency reads

$$E \left[|F(\omega)|^2 \right] = \int_0^L S(\omega, x) dx \quad (2)$$

Equations (1) and (2) are also known as the marginal spectral densities of a random field [2]. The power spectrum is called narrow-band, if the bulk of its energy is located only within a very small frequency band [11, 14]. It satisfies spectral separability, if it can be multiplicatively decomposed into a homogeneous spectrum $S(\omega)$ and a modulating envelope $g(x)$ in the form

$$S(\omega, x) = S(\omega) \cdot g(x) \quad (3)$$

If the power spectrum $S(\omega, x)$ of a random field is known, an arbitrary number m of corresponding random samples can be generated by the spectral representation method [21, 23], which reads for a one-dimensional univariate zero-mean Gaussian random field

$$f^{(i)}(x) = \sqrt{2} \sum_{n=0}^{N-1} A_n \cos \left(\omega_n x + \phi_n^{(i)} \right) \quad (4)$$

$$\text{with} \begin{cases} A_n = \sqrt{2 \cdot S(\omega_n, x) \cdot \Delta\omega} \\ \omega_n = n \cdot \Delta\omega \\ \Delta\omega = \omega_{up} / N \\ A_0 = 0 \quad \text{or} \quad S(\omega_0 = 0, x) = 0 \end{cases} \quad (5)$$

where $i = 1, 2, \dots, m$ and $n = 0, 1, 2, \dots, (N - 1)$. The parameter ω_{up} is the cut-off frequency, beyond which the power spectrum is assumed to be zero, the integer N determines the discretization of the active frequency range, and $\phi_n^{(i)}$ denotes the (i) -th realization of N independent phase angles uniformly distributed in the range $[0, 2\pi]$. For non-Gaussian random fields, the translation field theory can be used to generate random samples from a simple transformation of an underlying Gaussian field [5].

The spectrum estimation methods to be presented in the following are tested by a standard separable benchmark problem, the uniformly modulated Kanai–Tajimi spectrum [8, 22], which is defined in view of Eq. 3 by its separable components

$$S(\omega) = \frac{1 + 4\zeta^2 \left(\frac{\omega}{\omega_0}\right)^2}{\left[\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\zeta \frac{\omega}{\omega_0}\right)^2 \right]} \quad (6)$$

$$g(x) = \frac{e^{-0.25x} - e^{-0.5x}}{0.25} \quad (7)$$

Parameters $\omega_0 = 10$ rad/mm and $\zeta = 0.24$ represent the natural frequency and the damping ratio, respectively. Their specific values in conjunction with the exponential modulating function of Eq. 7 are adopted from [22] and yield a power spectrum with equally pronounced evolution in space and frequency directions, therefore representing a suitable benchmark for evolutionary estimation techniques (see Figs. 1 and 2).

The Kanai–Tajimi spectrum of Fig. 2 is used to generate 10,000 corresponding random field samples $f^{(i)}(x)$, $i = 1, 2, \dots, 10,000$ via the spectral representation formula Eq. 4. Estimates of the Kanai–Tajimi spectrum are obtained by substituting

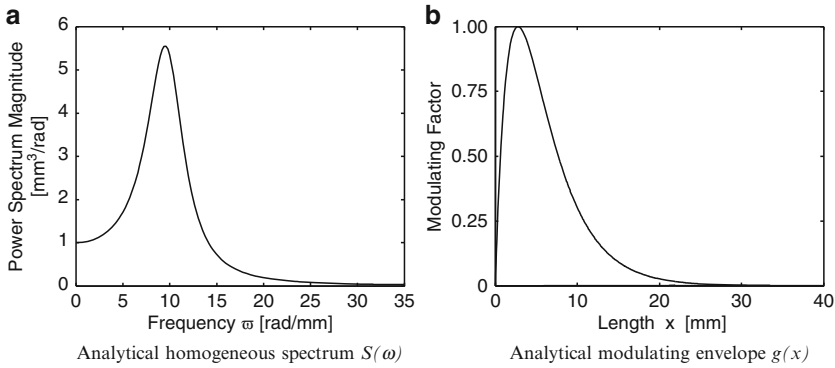


Fig. 1 Components of the separable Kanai–Tajimi benchmark spectrum

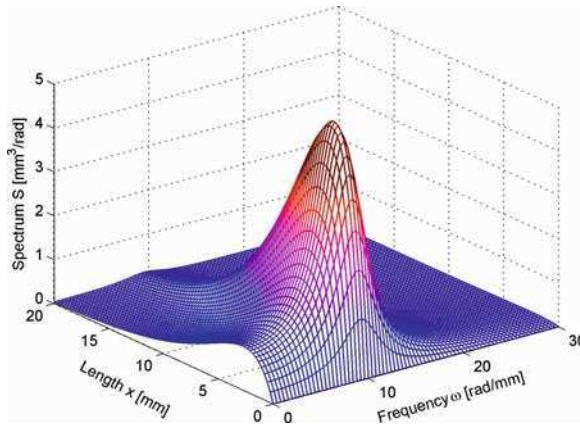


Fig. 2 Analytical reference solution of the separable Kanai–Tajimi spectrum

samples $f^{(i)}(x)$ into the estimation methods introduced in the following, whose results can be assessed by comparing them to the analytical reference of Fig. 2.

3 Existing Methods for Evolutionary Spectrum Estimation

The estimation of a homogeneous Fourier power spectrum is a standard method [11, 14], which can be obtained from a series of samples $f^{(i)}(x)$ by the so-called periodogram

$$\tilde{S}_h(\omega) = E \left[\frac{1}{2\pi L} \cdot \left| \int_0^L f^{(i)}(x) \cdot e^{-I\omega x} dx \right|^2 \right] \quad (8)$$

with L being the total sample length. The homogeneous spectrum estimate $\tilde{S}_h(\omega)$ of the evolutionary Kanai–Tajimi spectrum is shown in Fig. 3. Whereas the energy distribution in frequency direction is predicted correctly, the spatial location of the energy peak is lost, since the Fourier transform in Eq. 8 averages the energy variation in space over the whole length L . To preserve this information, samples $f^{(i)}(x)$ have to be transferred into evolutionary power spectra.

3.1 The Short-Time Fourier Transform

The most common approach for evolutionary power spectrum estimation is based on the short-time Fourier transform (STFT), also referred to as moving window or Gabor transform [1, 2]. The idea of STFT is to emphasize the samples $f^{(i)}(x)$ at

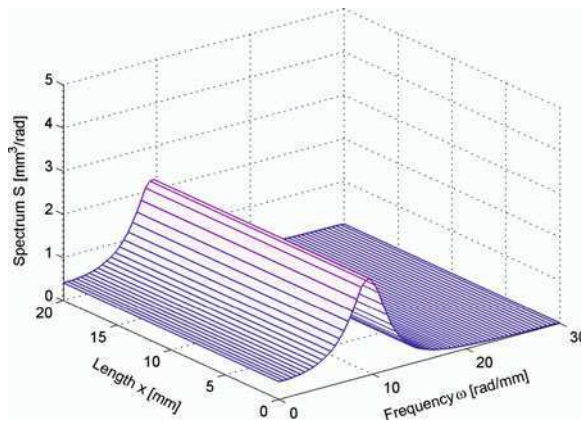


Fig. 3 Periodogram based estimate of the Kanai–Tajimi spectrum

a distinct spatial position $x = \chi$, whose local properties are to be studied, and to suppress them at positions farther away from χ . This is achieved by multiplying the samples with a window $w(x - \chi)$ of finite width T centered at the position of interest χ . The window is then moved along the spatial axis at n equally-spaced positions $x = \chi^j, j = 1, \dots, n$, and the Fourier transform is applied at each χ^j . This strategy results in homogeneous spectrum components $\tilde{S}_j(\omega, x)$ for each χ^j

$$\tilde{S}_j(\omega, x) = E \left[\frac{1}{2\pi T} \cdot \left| \int_{\chi-T/2}^{\chi+T/2} f^{(i)}(x) \cdot w(x - \chi^j) \cdot e^{-I\omega x} dx \right|^2 \right] \quad (9)$$

The complete evolutionary spectrum estimate, also known as the spectrogram, can finally be obtained by combining all components $j = 1, 2, \dots, N$. The basic limitation of the method is its inability to achieve simultaneous localization in both frequency and space, which is a consequence of the uncertainty principle [1, 2, 15]. If the spatial localization is increased by shortening the window width T , the frequency resolution of the spectrum deteriorates. In turn, if the width T is increased, frequencies are resolved better, but the spatial localization is reduced.

The result of the STFT based estimation for the Kanai–Tajimi benchmark spectrum is shown in Fig. 4 and is obtained with a simple non-overlapping rectangular window

$$w(x) = \begin{cases} 1 & -T/2 \leq x \leq T/2 \\ 0 & \text{elsewhere} \end{cases} \quad (10)$$

centered at 16 equally-spaced positions χ^j . A compromise in the spatial localization of the window function is chosen, that allows for a fair localization in space without distorting the frequency localization too severely.

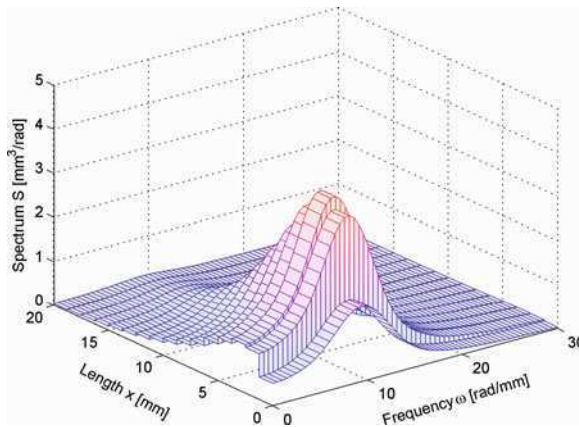


Fig. 4 STFT based estimate of the Kanai–Tajimi Spectrum

3.2 The Harmonic Wavelet Transform

For the joint space–frequency representation of evolutionary power spectra, harmonic wavelets developed by Newland [11] have proved to be especially suitable due to their exact box-like Fourier spectrum. Wavelet functions of different frequency scales (m, n) and positions k can be interpreted as a collection of window functions that are mutually orthogonal, so that a generalized wavelet transform of a series of samples $f^{(i)}(x)$ can be constructed, resulting in wavelet coefficients $a_{(n,m),k}^{(i)}$. Because of the strongly localized amplitudes of harmonic wavelets of the same frequency scale (m, n) at different positions $x - k/(n - m)$, corresponding wavelet coefficients $a_{(n,m),k}^{(i)}$ can be used to distinguish local events of samples $f^{(i)}(x)$ at the same frequency. On this basis, Spanos and co-workers [22] have recently shown that an evolutionary power spectrum can be estimated from the wavelet coefficients as

$$\tilde{S}_{(m,n),k} = \frac{4 E \left[\left| a_{(m,n),k}^{(i)} \right|^2 \right]}{n - m} \quad (11)$$

This formula defines the localized spectral density in the discrete space–frequency regions

$$\frac{m 2\pi}{L} \leq \omega < \frac{n 2\pi}{L} \quad \text{and} \quad \frac{kL}{n - m} \leq x < \frac{(k + 1)L}{n - m} \quad (12)$$

where integers $k = 0, \dots, (n - m - 1)$ determine the coupling between space and frequency localization. The major drawback of the harmonic wavelet based method is again the limitation of simultaneous space–frequency localization by Eqs. 12. They allow either for a fine resolution in frequency, if differences in scales (m, n) are small, or for a fine resolution in space, if differences in scales (m, n) are large. The wavelet based estimate for the Kanai–Tajimi benchmark is shown in Fig. 5, which shows reasonable localization properties due to a compromise in space–frequency resolution.

3.3 The Wigner–Ville Transform

A qualitatively different approach is provided by the Wigner–Ville transform [10], which yields the following evolutionary spectrum expression from a series of samples $f^{(i)}(x)$

$$\tilde{S}(\omega, x) = E \left[\frac{1}{2\pi} \cdot \int_0^L f^{(i)} \left(x + \frac{\tau}{2} \right) \cdot f^{(i)} \left(x - \frac{\tau}{2} \right) \cdot e^{-I \omega \tau} d\tau \right] \quad (13)$$

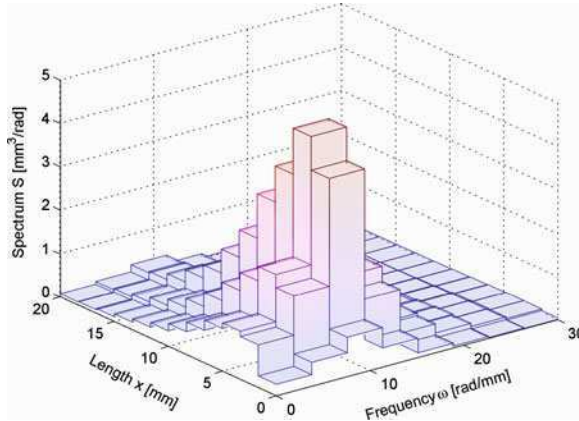


Fig. 5 Harmonic wavelet based estimate of the benchmark spectrum

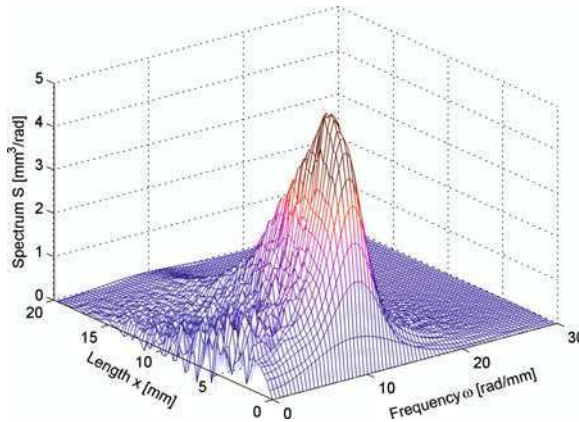


Fig. 6 Wigner–Ville based estimate of the Kanai–Tajimi spectrum

where τ is a shifting parameter. Equation 13 can be interpreted as a folding of the left over the right part of the sample to determine possible overlaps, which will be then present in the spectrum at the current position x [1]. In a more formal sense, Eq. 13 can be interpreted as the Fourier transform of an autocorrelation estimate [14].

The major strength of the Wigner–Ville based spectrum estimate is the exact representation of the marginal densities Eqs. 1 and 2, which guarantees an exact representation of the incremental energy content of the spectrum. Its major drawback is its potential ability to yield negative spectral values, which contradicts the mathematical definition of the power spectrum and its physical energy interpretation [1, 2]. The Wigner–Ville estimate of the Kanai–Tajimi benchmark spectrum is shown in Fig. 6.

4 The Method of Separation: Evolutionary Spectrum Estimation of Separable Random Fields

The method of separation is generally valid for the estimation of separable power spectra, but in particular designed to cope with the challenge of accurate simultaneous space–frequency localization. For the estimation of non-separable power spectra, it can be complemented by a joint strategy, which is based on the partitioning of the space–frequency domain into several sub-spectra that have to be separable only within themselves. These sub-spectra can then be estimated consecutively by the method of separation and finally be recomposed to the full non-separable spectrum. For the introduction of the principles of the method, the present text confines itself to the separable case, details on the estimation of non-separable spectra with the method of separation can be found in Schillinger and Papadopoulos [19].

4.1 Theory and Derivation

The method of separation assumes that input samples $f^{(i)}(x)$ represent a separable or at least approximately separable random field. The essential advantage of this assumption is the breakdown of the combined evolutionary spectrum estimation into a frequency and a spatial part, which can be dealt with separately. The definition of spectral separability in Eq. 3 allows that its multiplicative components can be chosen arbitrarily from a group of pairs $[S'; g']$ that satisfy with respect to the original components $[S; g]$

$$\begin{aligned} S'(\omega) &= \lambda \cdot S(\omega) \\ g'(x) &= \frac{1}{\lambda} \cdot g(x) \end{aligned} \quad (14)$$

where λ is an arbitrary positive number. Equation (14) constitutes sets of geometrically similar functions, whose energy content is varied by λ , but whose energy distribution over frequency or space, i.e. the relative shapes of the curves, remain the same. The product of Eq. 14 yields always the correct evolutionary spectrum $S(\omega, x)$ in the sense of Eq. 3.

In the method of separation, the spectrum component S' of Eq. 14 is chosen as the homogeneous Fourier power spectrum

$$S_h(\omega) = \frac{1}{L} \cdot \int_0^L S(\omega, x) dx = \lambda_h \cdot S(\omega) \quad (15)$$

corresponding to the frequency content of the evolutionary spectrum $S(\omega, x)$ averaged over the signal length L . Its counterpart $g_h(x)$ can be obtained from the first expression of Eqs. 14 by factor

$$\lambda_h = \frac{1}{L} \cdot \int_0^L g(x) dx \quad (16)$$

The evolutionary spectrum $S(\omega, x)$ in the method of separation is thus decomposed into

$$S(\omega, x) = S_h(\omega) \cdot g_h(x) \quad (17)$$

An estimate $\tilde{S}_h(\omega)$ of Eq. 15 can be readily obtained from the periodogram Eq. 8, where the frequency content of the separable input samples are averaged over their length L by the Fourier transform [1, 2], which corresponds to the definition of $S_h(\omega)$ in Eq. 15. An estimate $\tilde{g}_h(x)$ for the spatial envelope can be derived from the mean square of samples $f^{(i)}(x)$. The basic analytical expression is found from Eq. 1 as

$$\begin{aligned} E[|f(x)|^2] &= 2 \int_0^\infty S_h(\omega) g_h(x) d\omega \\ &= g_h(x) \cdot 2 \int_0^\infty S_h(\omega) d\omega \end{aligned} \quad (18)$$

The estimate $\tilde{g}_h(x)$ is obtained by replacing in Eq. 18 the analytical homogeneous Fourier spectrum $S_h(\omega)$ and the mean square $E[|f(x)|^2]$ by corresponding estimates, which yields

$$\tilde{g}_h(x) = \frac{E[|f^{(i)}(x)|^2]}{2 \int_0^\infty \tilde{S}_h(\omega) d\omega} \quad (19)$$

The final estimate of the spectrum can now be established by replacing the analytical expressions in Eq. 18 by their estimates $\tilde{S}_h(\omega)$ of Eq. 9 and $\tilde{g}_h(x)$ of Eq. 18, which results in

$$\tilde{S}(\omega, x) = E[|f^{(i)}(x)|^2] \cdot \frac{\tilde{S}_h(\omega)}{2 \int_0^\infty \tilde{S}_h(\omega) d\omega} \quad (20)$$

Factor 1/2 in the right-hand side fraction is necessary, because Eq. 20 takes into account only one side of the symmetric two-sided power spectrum.

The validity of Eq. 20 can be succinctly illustrated by energy considerations. The right-hand side fraction contains the estimate $\tilde{S}_h(\omega)$ in the numerator, which is normalized by the denominator in the sense that its total energy content, i.e. the area under the two-sided curve $\tilde{S}_h(\omega)$, is a constant of 1. This can be trivially verified by integration of the right-hand side fraction in Eq. 20 over frequency as

$$\int_0^{\infty} \frac{\tilde{S}_h(\omega)}{\int_0^{\infty} \tilde{S}_h(\omega) d\omega} = 1 \quad (21)$$

The left-hand side of Eq. 20 represents the mean square of the samples $f^{(i)}(x)$, which on the basis of Eq. 2 is an estimation of the incremental energy in space. Hence, Eq. 21 can be interpreted as the distribution of the mean square over the frequency domain by a normalized homogeneous spectrum for each position x . This makes the method of separation a direct implementation of the initial intuitive concept of the power spectrum.

4.2 Performance Test with Kanai–Tajimi Benchmark Estimation

The complete Kanai–Tajimi spectrum estimate obtained with the method of separation is shown in Fig. 7. The method of separation captures exact spectrum gradients and peak values as seen in Fig. 2 considerably better than STFT and wavelet transforms of Figs. 4 and 5 and leads to a more regular spectrum surface than the Wigner–Ville transform of Fig. 6.

Stochastic convergence in a Monte-Carlo sense [14] is tested by the squared difference between exact analytical and estimated Kanai–Tajimi spectra S_{ex} and $\tilde{S}_m$, respectively, integrated over the space–frequency domain in the form

$$e(m) = \int_0^{\infty} \int_0^L (S_{ex} - \tilde{S}_m) dx d\omega \quad (22)$$

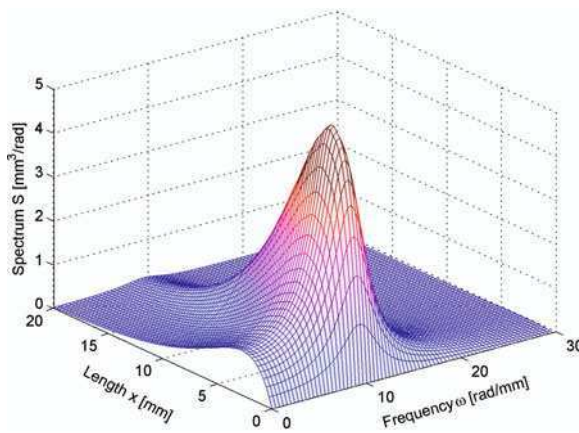


Fig. 7 Method of separation based estimate of the Kanai–Tajimi spectrum

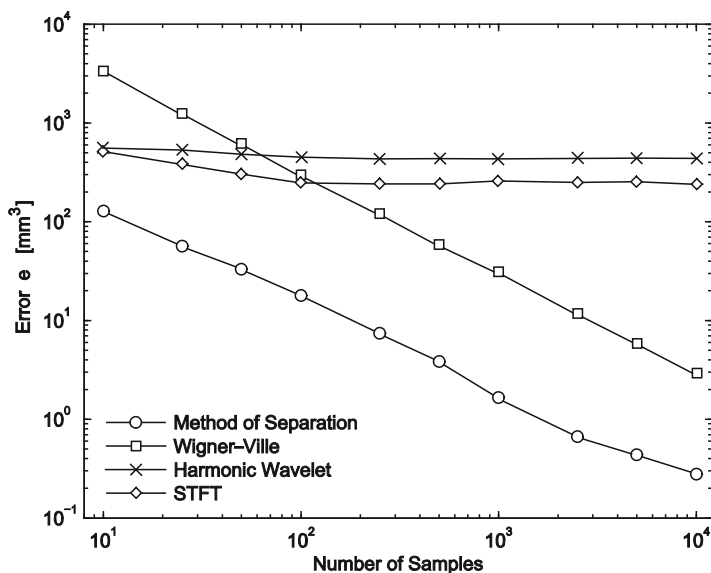


Fig. 8 Convergence behaviour of the examined estimation techniques for the Kanai-Tajimi benchmark

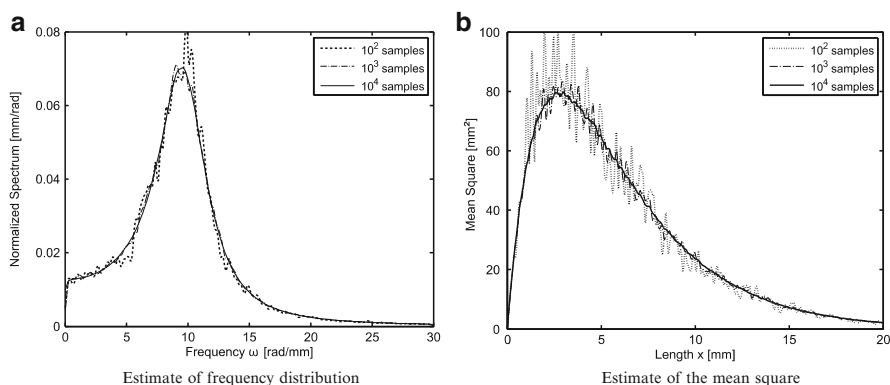


Fig. 9 Method of separation based estimates of the frequency and spatial component of the Kanai-Tajimi spectrum with different sample sizes

The error e of Eq. 22, which depends on the number m of input samples $f^{(i)}(x)$, $i = 1, \dots, m$ used for the computation of $\tilde{S}_m$, is plotted in Fig. 8 for each estimation method. Whereas STFT and wavelet estimates do not converge due to the systematic error of the uncertainty principle, the method of separation achieves monotonic convergence, approaching the exact solution faster than the Wigner-Ville method.

The two principal components the method of separation, i.e. the estimated mean square (left-hand side of Eq. 20) and the estimated normalized frequency distribution (right-hand side of Eq. 20), are shown in Fig. 9a and b, respectively.

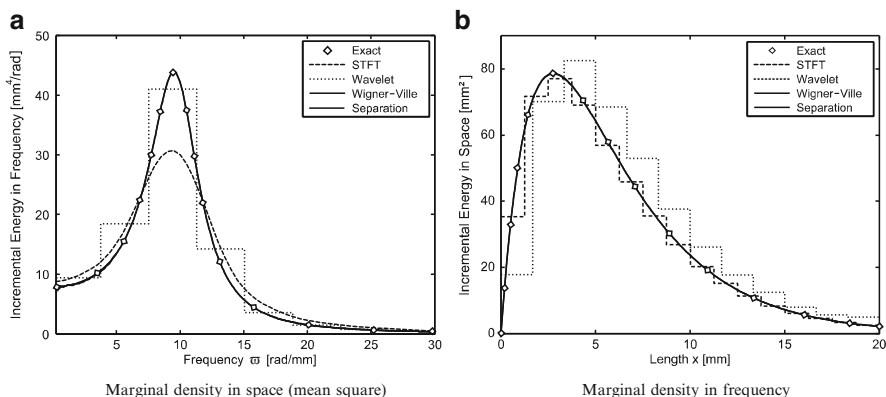


Fig. 10 Marginal densities (incremental energies) of the Kanai–Tajimi spectrum estimates

The geometric similarity of the component estimates in Fig. 9 with respect to the analytical components in Fig. 1 illustrates the validity of the fundamental expressions in Eq. 14. The marginal densities Eqs. 1 and 2, which are an excellent indicator for the quality of space and frequency localization, respectively [2], are plotted in Fig. 10. The method of separation and the Wigner–Ville transform are able to reproduce the analytical incremental energies exactly due to their accurate localization properties in both space and frequency directions, whereas STFT and harmonic wavelet based methods exhibit observable deviations from the exact solutions as a consequence of the uncertainty principle.

In summary, it can be stated that the method of separation based estimate of the Kanai–Tajimi spectrum is more accurate in terms of regularity of the spectrum surface, space–frequency localization and convergence towards the exact spectrum than the results of the established estimation techniques.

5 Stochastic Modeling of Imperfections in Structures: Estimation of Strongly Narrow-Band Power Spectra

The introduced space–frequency analysis techniques are now applied to a practical problem from stochastic imperfection modeling in structures. The target power spectrum represents the stochastic variation of geometric imperfections in an I-section flange and has to be estimated from corresponding experimental measurements obtained from six nominally equal I-section flanges. The measured geometric imperfections represent the deviation of the true flange edge position from perfect plate geometry along the beam axis as illustrated in Fig. 11. The zero-mean parts of the imperfection measurements, which have been previously separated from their deterministic mean $\mu(x)$, represent six input samples $f^{(i)}(x)$, $i = 1, 2, \dots, 6$, which are plotted in Fig. 12a. Before the presentation of the estimation results,

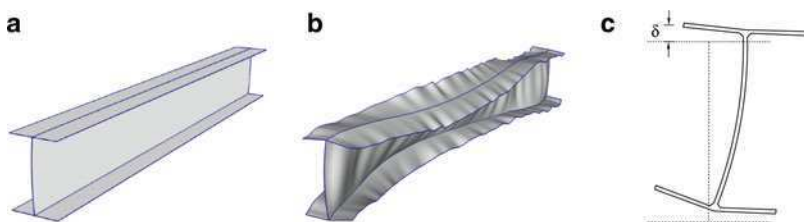


Fig. 11 The I-section beam experimentally investigated by Hasham and Rasmussen [6]

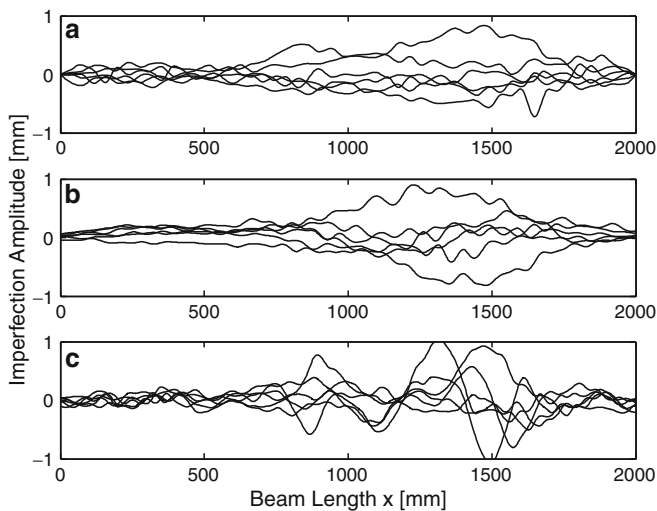


Fig. 12 Imperfection samples: (a) measurements; (b) method of separation based simulation; (c) STFT based simulation

some key issues related to the space–frequency analysis of strongly narrow-band random functions are discussed on the basis of the imperfection example. The complete results of the stochastic buckling simulation for the I-section beam-column corresponding to the present imperfection example can be found in [20].

5.1 Spectral Separability

The assumption of spectral separability in the method of separation introduces an error in case of non-separable input samples. Non-separability implies that the energy distribution over the frequency domain, i.e. the spectrum shape in frequency direction, differs along the length x . In the case of the strongly narrow-band imperfection measurements, possible variations of energy distribution in frequency

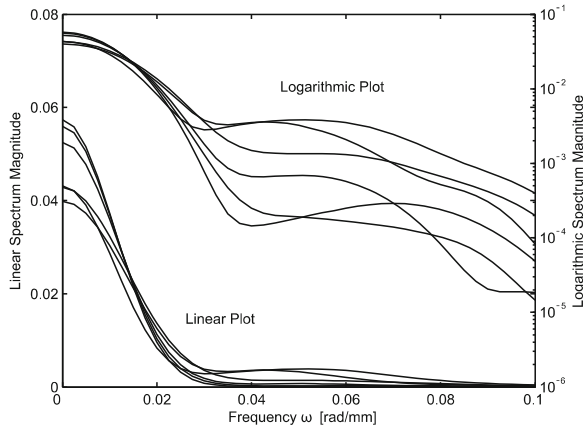


Fig. 13 Normalized STFT based spectrum components

are considerably limited, since the main lobe, which contains the largest part of the energy, is located only within a very small fixed bandwidth between 0 and 0.005 rad/mm. This is illustrated by a STFT based spectrum estimate according to Eq. 9, obtained at six equally spaced positions χ^j by a non-overlapping Hamming window [1, 2, 9] of length $L/6$. The six individual STFT components plotted in Fig. 13 are normalized with respect to the total energy represented by the area under their curves. The discrepancy between the plotted curves with respect to shape and amplitude thus indicates the degree of non-separability. Within the narrow-band main lobe containing about 95% of the total energy, approximate spectral separability can be readily verified, since the six curves exhibit the same qualitative shape and diverge only slightly.

5.2 Number of Samples and Spectral Smoothing

In most practical applications, only a very limited number of measurements are available. The resulting insufficient ensemble averaging in the evaluation of the operator $E[\]$ in Eq. 20 leads to spurious oscillations in the evolutionary spectrum estimate, due to localized under- and overestimation of the true spectrum. The spurious oscillations are stronger in spatial direction x , because the evaluation of the homogeneous spectrum component Eq. 8 inherently implies an additional averaging of frequency content over length L . An effective damping of the spurious oscillations in spatial direction can be accomplished by spectral smoothing

$$\hat{S}(\omega, x_k) = \frac{1}{2n+1} \sum_{m=-n}^n \tilde{S}(\omega, x_{k+m}) \quad (23)$$

where x_k denote sample points in space of the discrete spectrum representation and $\hat{S}$ is the smoothed spectrum counterpart. Equation (23) can be imagined as a window, which is moved in small steps along the spatial axis, successively replacing the central spectrum values by the arithmetic average of all visible values. The empirical window size $(2n + 1)$ has to be chosen small enough not to distort the evolutionary trend, but large enough to effectively smoothen the spurious local oscillations.

5.3 The Uncertainty Principle

Originally motivated from a physical point of view by Heisenberg's observations in quantum mechanics, the uncertainty principle has turned out to be a general property of Fourier analysis [1, 2, 15]. It states that the two components of a Fourier transform pair, such as a geometric imperfection sample $f(x)$ and its Fourier transform $F(\omega)$, cannot be completely localized at the same time. The implication of the uncertainty principle can be illustrated by an increase of localization in the mean square $E[|f^{(i)}(x)|^2]$ as illustrated in Fig. 14a, which is obtained by applying Hamming windows [1, 2, 9] of decreasing widths to measurements $f^{(i)}(x)$. Shortening the window width T , however, necessitates a spread of the corresponding homogeneous spectrum estimate $\tilde{S}_h(\omega)$ Eq. 8 as shown in Fig. 14b. Due to their small frequency bandwidths, strongly narrow-band spectra are especially sensitive to this phenomenon.

5.4 Performance of Available Estimation Techniques

The spectrum estimates of the corresponding imperfection example obtained from the wavelet, Wigner–Ville, and STFT based techniques and the method of separation are shown in Fig. 15. All measurements have been processed by a Hamming window prior to the evaluation of Fourier transforms to minimize spectral leakage [9]. The harmonic wavelet based estimate in Fig. 15a exhibits satisfactory localization in frequency direction, which can be obtained by the smallest possible difference in scales $n - m = 1$. However, this choice leads to the complete loss of localization in space according to Eq. 12. The Wigner–Ville estimate in Fig. 14b can be expected to give an accurately localized impression of the true power spectrum. However, its major disadvantage is the appearance of negative spectral density over large parts of the space–frequency domain. The harmonic wavelet based method and the Wigner–Ville method thus yield estimates, which are unsuitable to be used in the framework of spectral representation. The STFT based estimate in Fig. 15c, which has been obtained with 16 overlapping Hamming windows [1, 2, 9] of width $L/8$, exhibits good localization in space. However, in comparison to the Wigner–Ville and method of separation results, which rely on a Fourier transform with full window width $T = L$,

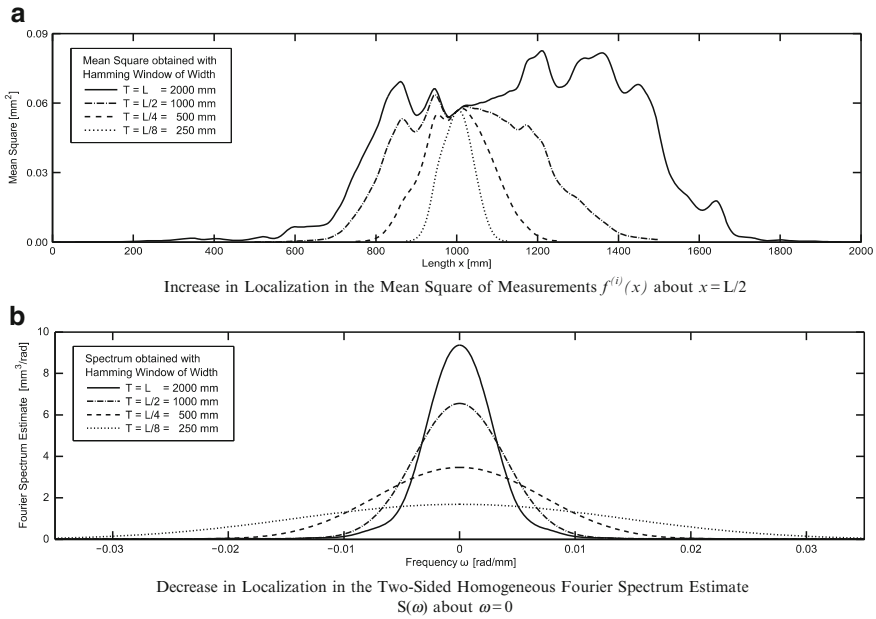


Fig. 14 The Uncertainty Principle: Basic Mechanism for the Imperfection Example

the spread of the main lobe of the STFT-based estimate is dramatically increased due to the uncertainty principle (observe the frequency scale).

The accurately localized result of the method of separation is shown in Fig. 15d. Due to the small number of only six available measurements, spectral smoothing Eq. 23 is applied to the initial estimate in spatial direction, which results in the final smoothed spectrum estimate Fig. 15e. In view of the uncertainty principle, the frequency localization in the Fourier transform of a strongly narrow-band function corresponds to the availability of a sufficiently long signal length L . Since the method of separation Eq. 20 allows the use of the full available length L in the Fourier transforms involved, its spectrum estimate guarantees an optimum frequency localization and a minimum spread of the energy distribution. The accurate simultaneous space–frequency localization in the method of separation based estimate can be further evidenced by its marginal densities Eqs. 1 and 2 in Fig. 16. The qualitative accordance of its energy distribution and peak values with the Wigner–Ville estimate Fig. 15b additionally supports its plausibility. Finally, the smoothed spectrum estimate of the method of separation and the STFT based spectrum estimate are used to generate six random imperfection samples by spectral representation Eq. 4, which are plotted in Fig. 12b and c. As a result of the large frequency spread in the STFT based spectrum, the wave-lengths of the method of separation based samples correlate considerably better with the measured counterparts of Fig. 12a than the STFT based samples.

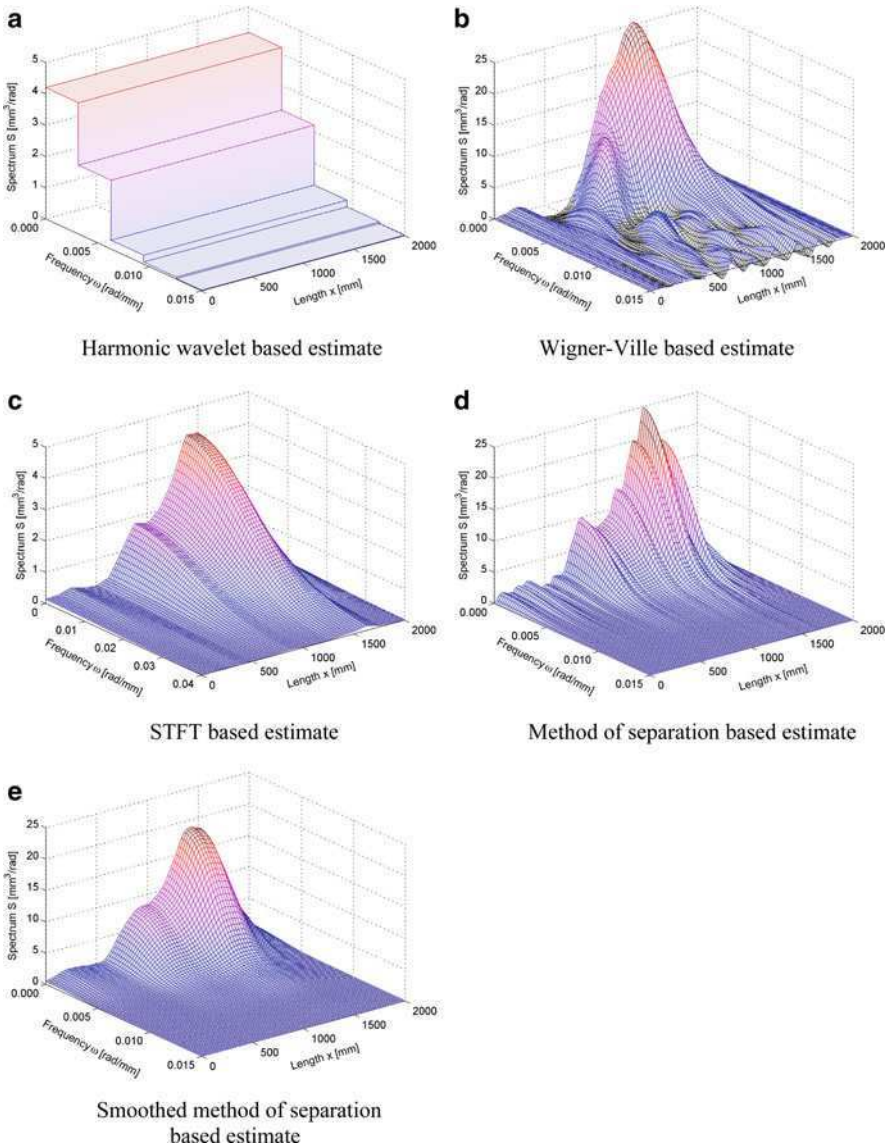


Fig. 15 Evolutionary power spectrum estimates for the narrow-band imperfection example

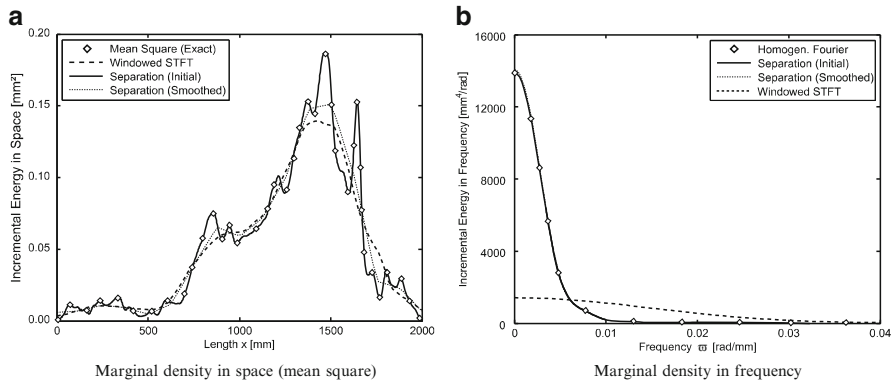


Fig. 16 Marginal densities (incremental energies) of STFT and method of separation based spectrum estimates of the imperfection example

6 Summary and Conclusions

The present paper deals with space–frequency analysis techniques for the estimation of evolutionary power spectra, which are used in engineering practice for the accurate stochastic simulation of random physical phenomena by spectral representation. In addition to established techniques, a method for the estimation of separable power spectra is introduced, called method of separation, which yields accurate simultaneous space–frequency localization. For the estimation of the separable Kanai–Tajimi benchmark spectrum, the method of separation based estimate is demonstrated to be more accurate in terms of surface regularity, space–frequency localization and stochastic convergence than estimates from the established techniques. It is furthermore illustrated by an example from imperfection modeling in structures that the method of separation yields good localization in space and frequency even in for the case of strong narrow-bandedness of the target spectrum, where established space–frequency analysis techniques fail to provide acceptable estimation results.

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References

1. Carmona, R., Hwang, W.J., Torrésani, B.: Practical Time–Frequency Analysis. Academic Press, New York (1998)
2. Cohen, L.: Time–Frequency Analysis. Prentice-Hall, Upper Saddle River, NJ (1995)

3. Conte, J.P., Peng, B.F.: Fully nonstationary analytical earthquake ground-motion model. *J. Eng. Mech. ASCE* **123**, 15–24 (1997)
4. Deodatis, G., Spanos, P.D. (eds.): Computational stochastic mechanics. In: Proceedings of the 5th International Conference on Computational Stochastic Mechanics (CSM-5), IOS Press, Amsterdam (2006)
5. Grigoriu, M.: Simulation of stationary non-Gaussian translation processes. *J. Eng. Mech. ASCE* **124**(2), 121–126 (1998)
6. Hasham, A.S., Rasmussen, K.J.R.: Member capacity of thin-walled I-sections in combined compression and major axis bending. Research Report No R746, School of Civil Engineering, University of Sydney (1997)
7. Liang, J., Chaudhuri, S.R., Shinozuka, M.: Simulation of non-stationary stochastic processes by spectral representation. *J. Eng. Mech. ASCE* **133**, 616–627 (2007)
8. Lin, Y.K., Yong, Y.: Evolutionary Kanai–Tajimi earthquake models. *J. Eng. Mech. ASCE* **113**, 1119–1137 (1987)
9. Lyons, R.G.: Understanding Digital Signal Processing. Prentice Hall, Upper Saddle River, NJ (2004)
10. Martin, W., Flandrin, P.: Wigner–Ville spectral analysis of non-stationary processes. *IEEE Trans. Acoust. Speech Signal Process* **33**(6), 1461–1470 (1985)
11. Newland, D.E.: An Introduction to Random Vibrations, Spectral & Wavelet Analysis. Wiley, New York (1993)
12. Papadopoulos, V., Papadrakakis, M.: The effect of material and thickness variability on the buckling load of shells with random initial imperfections. *Comput. Methods Appl. Mech. Eng.* **194**, 1405–1426 (2005)
13. Papadopoulos, V., Charnpis, D.C., Papadrakakis, M.: A computationally efficient method for the buckling analysis of shells with stochastic imperfections. *Comput. Mech.* **43**, 687–700 (2009)
14. Papoulis, A., Pillai, S.U.: Probability. Random Variables and Stochastic Processes. McGraw-Hill, New York (2002)
15. Pinsky, M.: Introduction to Fourier Analysis and Wavelets. American Mathematical Society, Providence, RI (2009)
16. Priestley, M.B.: Spectral Analysis and Time Series. Academic, London (1981)
17. Priestley, M.B.: Nonlinear and Non-stationary Time Series Analysis. Academic, London (1988)
18. Schenk, C.A., Schuëller, G.I.: Buckling analysis of cylindrical shells with cutouts including random boundary and geometric imperfections. *Comput. Methods Appl. Mech. Eng.* **196**: 3424–3434 (2007)
19. Schillinger, D., Papadopoulos, V.: Accurate estimation of evolutionary power spectra for strongly narrow-band random fields. *Comput. Methods Appl. Mech. Eng.* **199**(17–20), 947–960 (2010)
20. Schillinger, D., Papadopoulos, V., Bischoff, M., Papadrakakis, M.: Buckling analysis of imperfect I-section beam-columns with stochastic shell finite elements. *Comp. Mech.* **46**(3), 495–510, 2010.
21. Shinozuka, M., Deodatis, G.: Simulation of stochastic processes by spectral representation. *Appl. Mech. Rev. ASME* **44**, 191–203 (1991)
22. Spanos, P.D., Tezcan, J., Tratskas, P.: Stochastic processes evolutionary spectrum estimation via harmonic wavelets. *Comput. Methods Appl. Mech. Eng.* **194**, 1367–1383 (2005)
23. Stefanou, G., Papadrakakis, M.: Assessment of spectral representation and Karhunen–Loève expansion methods for the simulation of Gaussian stochastic fields. *Comput. Methods Appl. Mech. Eng.* **196**, 2465–2477 (2007)
24. Stefanou, G.: The stochastic finite element method: past, present and future. *Comput. Methods Appl. Mech. Eng.* **198**, 1031–1051 (2009)
25. Vanmarcke, E.: Random Fields: Analysis and Synthesis. MIT Press, Cambridge, MA (1983)

A Seismologically Consistent Husid Envelope Function for the Stochastic Simulation of Earthquake Ground-Motions

Sara Sgobba, Peter J. Stafford, and Giuseppe C. Marano

Abstract Earthquake-induced ground-motion may be realistically described as random processes that are intrinsically non-stationary in both amplitude and frequency content. In order to take into account the finite duration and the amplitude non-stationarity of earthquake-induced ground-motions, it is common practice to modulate a stationary stochastic process with a shaping window, or envelope function, in order to obtain a transient signal. Different shapes have been proposed in literature for such windows but the main problem until now has been how to correlate the parameters of the shaping window to the characteristics of some seismic design scenario (such as magnitude, distance, and site conditions). In this work, an envelope function is proposed that makes use of parameters that are commonly available in seismic design situations involving a scenario-type analysis. Such a scenario may be deterministically prescribed, or may be the result of a probabilistic seismic hazard analysis. The envelope function is also directly related to the Arias intensity of the ground motion and has a functional form closely related to that of a lognormal probability density function. The envelope function may be used, in conjunction with suitable peak factors, to predict the distribution of peak ground acceleration values corresponding to a given earthquake scenario.

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1 Introduction

Visual inspection of earthquake-induced ground accelerations reveal an intrinsically random nature. The randomness of the ground motion arises as a result of the processes of seismogenesis and wave propagation being highly nonlinear. While state-of-the-art approaches to deterministic modelling of ground-motions using seismological theory are impressive and account for many phenomena, the degree of natural variability that is observed among motions from similar seismological scenarios is so great that it is not possible to confidently prescribe future motions using such deterministic approaches. Nevertheless, while observed accelerograms appear random when viewed individually, groups of records from similar scenarios do exhibit common features. Therefore, it is clear that a suitable approach to modelling future accelerograms is provided by methodologies that are typical of probabilistic and stochastic mechanics. A robust way to treat these phenomena is to employ Random Vibration Theory (RVT), which is a widely applied mathematical tool in other similar engineering problems where dynamic actions exhibit an analogous random nature, such as in offshore and aeronautical engineering.

Random process representations of earthquake-induced ground-shaking can be employed for both generating simulated acceleration time-histories and in pure random vibration analysis, in order to obtain structural responses and safety evaluations using well-established theory [15]. In both cases it is clear that the proper modelling of these processes plays a central role in the accuracy of the analysis. Hence, in earthquake engineering, the proper consideration of the evolutionary nature of real seismic records is indispensable. For these reasons, an appropriate representation of ground acceleration must be based upon a random process with time-dependent amplitude and frequency content. Both of these non-stationarities can strongly influence structural responses and safety evaluations.

Within the context of modelling seismic actions as random process, the first attempt was made by using a simple stationary model [6, 14]. These early models generated time histories that represented a stationary stochastic Gaussian process of theoretically infinite duration. However, it has since been recognised that these models are unable to preserve the real time-varying amplitude characteristics of natural earthquake accelerograms. Following the introduction of the concept of evolutionary power spectrum [18] some non-stationary stochastic models have been proposed, including the modulated stationary process [7] and the filtered modulated stationary process [11]. Some years later, Lin and Yong [13] developed the evolutionary Kanai-Tajimi [10, 28] earthquake models as a non-stationary random pulse. Afterwards, Fan and Ahmadi [5] expanded the site-dependent stationary Kanai-Tajimi model to non-stationary models, taking into account both time-varying amplitude and frequency content. Rezaeian and Der Kiureghian [21] have recently reviewed and discussed the most common of the adopted approaches in the literature for the stochastic simulation of ground motions.

Despite the broad awareness that accelerograms are characterised by significant amplitude non-stationarity and, in most cases, by a marked variability of the frequency content, the simplest and most commonly adopted hypothesis consists

of modelling ground motions as the product of a stationary signal and a deterministic modulation function in order to obtain a uniformly modulated process (uniformly modulated in frequency). However, the analysis of a large number of seismic records has shown that this hypothesis cannot always be adopted as it supposes constant frequency content throughout the ground motion. The temporal variation of the frequency content can, in fact, have a significant effect on estimates of structural response and on reliability evaluation. In such cases, it is necessary to develop a more complicated non-stationary model with time-varying frequency content. However, it is generally acknowledged that, for many applications, the component of the process that dominates amplitude-based measures of structural response, such as inter-storey drift, i.e., the *strong-shaking phase*, may be represented by a weakly stationary process [24]. For this reason, the frequency non-stationarity is often tactically neglected as it greatly simplifies the resulting mathematical operations.

When a filtered, uniformly modulated, non-stationary stochastic process is adopted for modelling seismic actions, the selection of the most suitable modulation function becomes a crucial issue. The modulating function must be able to reproduce the shape of real records and reflect the temporal distribution of energy and the strong-motion duration. Not a great deal of research has been conducted on the issue of how to most appropriately model the amplitude modulation, as random vibration methods have not been as widely applied to real-world seismic analyses as other empirically-based techniques. However, different forms have been proposed for the shaping window over the years with the aim of best representing the time-evolution of accelerograms (trapezoidal, double exponential, lognormal, etc.). Note that most envelope functions commonly used in earthquake engineering include: the step function used by Caughey and Stumpf [2], the boxcar function used by Barnoski and Maurer [1], the exponential function first used by Shinozuka and Sato [23], the product of polynomial and exponential functions used by Iyengar and Iyengar [7], Saragoni and Hart [22] and the piecewise expression used by Jennings et al. [9]. These functions have simple parametric forms and were assumed to be representative of the time variation of the amplitude, or intensity, of ground shaking, thereby modelling the rate of build-up, rate of decay, maximum intensity and strong-motion duration of the accelerograms. These authors employed a continuous-time formulation of the models and, with the exception of Saragoni and Hart [22], divide each accelerogram into three segments and model the frequency content of each segment by stationary processes. However, the arbitrary division into segments and the abrupt change in the frequency content from segment to segment is not very satisfactory for simulating strong ground-motion.

The primary purpose of the present work is to advance the current situation by offering a new envelope function that represents the first component of a full non-stationary stochastic approach to accelerogram generation. The approach taken in the present study is to define a new envelope function that could be correlated to the main characteristics of the expected ground-motion. For any scenario-based analysis, one should always be able to infer a magnitude, a source-to-site distance, and some characteristics regarding the site conditions in addition to the corresponding amplitudes of response spectral ordinates. The assumption that such information is available is pivotal to the methodology that is presented herein.

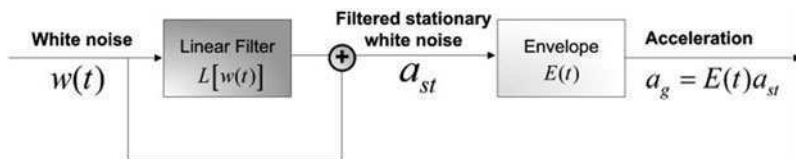


Fig. 1 Typical process for stochastic ground-motion simulation

The most common approach to simulating earthquake accelerograms consists of starting with a Gaussian stationary white noise process $w(t)$ which passes through a linear filter in order to obtain the desired power spectral density function and the result is then multiplied by a deterministic envelope function $E(t)$ so that the stationary filtered white noise a_{st} gets the desired time-dependent amplitude (Fig. 1).

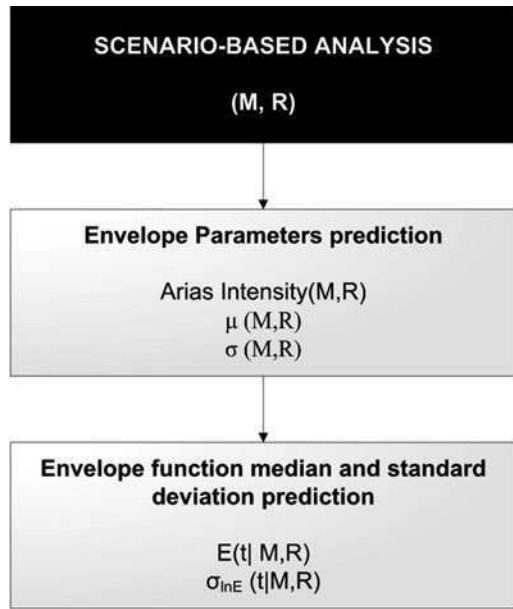
In Fig. 1 the specification of the frequency content of the ground motion is made through the generic linear filter, $L[\]$, but in most stochastic analyses the frequency content is prescribed directly in terms of the power spectral density function (PSDF) of the process. The simulation of accelerograms using this power spectral representation has recently been outlined rigorously by Liang et al. [12].

As previously mentioned, various different envelope functions have been proposed over the past 40 years or so; the most common of which are discussed by Jangid [8]. Quek et al. [19] suggest that, so long as the envelope function is able to appropriately reflect the energy content of the ground shaking, estimates of structural response are not particularly sensitive to the actual window function used. However, these authors were only concerned with the estimation of pseudo-spectral velocities and for other measures of structural response these findings may not hold [26]. Even if measures of the structural response are relatively insensitive to the windowing function it is still preferable to make this component of the stochastic model as realistic as possible. Therefore, the objective herein is to present an envelope function that is as realistic as possible while also ensuring that the energetic characteristics of the accelerogram are incorporated in a quantitative manner.

In this work an analytical expression for an envelope function is presented. As opposed to previously proposed formulations, the envelope is directly related to seismological and ground-motion parameters of specific earthquake scenarios that could be considered for design. The mathematical formulation is tied directly to the expected Arias intensity of the ground motion given some seismological scenario as well as to two shape parameters that are in turn related to seismological characteristics.

The information that is required in order to derive the envelope therefore consists of information regarding the scenario, i.e., magnitude, distance, average shear-wave velocity, etc., as well as an estimate of the Arias intensity associated with this particular combination of seismological parameters. This information may be specified directly from a scenario-based analysis, where one simply defines the seismological parameters from the outset. Figure 2 provides a schematic overview of the avenues via which one may obtain the information required to implement the envelope function that is presented herein.

Fig. 2 Schematic overview of the process via which one obtains an envelope function, $E(t)$, and associated (logarithmic) standard deviation, $\sigma_{\ln E(t)}$, for the scenario based analysis. The parameters μ and σ are generic envelope shape parameters that will be defined in what follows



In this way the envelope function is able to represent a specific seismic event, fully characterised by the seismological parameters that are commonly available in design situations. This is a key point of the procedure, because, in many previously-proposed approaches in stochastic dynamics, the envelope function is described by analytical formulations with a limited (or no) relation to any seismic scenario.

2 Mathematical Formulation of the Envelope Function

The form of the envelope function comes from the realisation that the Husid function $H(t)$:

$$H(t) = \frac{\int_0^t a^2(t)dt}{\int_0^\infty a^2(t)dt} = \frac{\pi}{2gI_a} \int_0^t a^2(t)dt \quad (1)$$

satisfies all of the required conditions to allow it to be regarded as a cumulative probability distribution function, as already shown by these authors [26]. Indeed, the shape of $H(t)$ for real accelerograms is strikingly similar to that of the cumulative distribution function (CDF) of the lognormal distribution (see Fig. 3).

This realisation leads one to suspect that a suitable candidate shape for an envelope function for accelerograms may be closely related to the probability density function equivalent of the normalized Husid function which is defined as in Eq. 2.

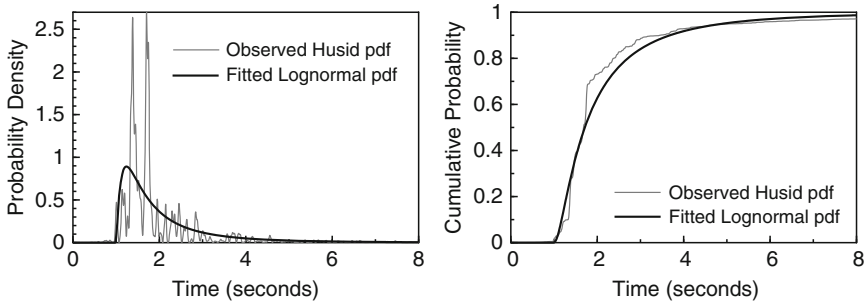


Fig. 3 Example of the typical fit that may be achieved by modelling the observed Husid pdf as a lognormal distribution [26]

$$h(t) = \frac{dH(t)}{dt} = \frac{\pi}{2gI_a} a^2(t) \quad (2)$$

In order to derive the form of the envelope function we simply assume that the function $h(t)$ may be modelled using the probability density function (pdf) of the lognormal distribution [26]. This lognormal pdf is directly related to the shape of the envelope and requires the specification of two parameters that correspond to the mean and standard deviation of the distribution. In this study we define these shape parameters as functions of common seismological parameters. Given the shape of the envelope, we must then scale this generic shape to ensure that the absolute amplitude of the envelope is consistent with the amplitudes that we would expect for a particular scenario [26].

2.1 Strong-Motion Dataset

The accelerograms that have been used in this study consist of records from the strong-motion database of the NGA project [3]. Not all of the available records have been considered in this study with the particular subset that has been adopted being exactly the same as that which has previously been used by Stafford and Bommer [25]. Here, it suffices to say that 2,406 records (4,812 individual horizontal components) from 114 earthquakes were included in the dataset for the present study. These records are associated with earthquakes having moment magnitudes ranging from 4.79 to 7.9 and depths to the top of their rupture ranging from 0 to 14.5 km. The recordings from these earthquakes are made at sites with average shear-wave velocities over the upper 30 m between 116 and 2,017 m/s and at locations ranging from 0.07 to 100 km from the assumed rupture surface of the events.

2.2 Calibration of the Model

In order to derive relationships to define the shape parameters, as well as to ensure that we achieve the correct amplitudes for the envelope, we must make direct comparisons with the envelopes of real accelerograms.

The calculation of the envelope of an accelerogram using the approach of Dugundji [4] consists of two steps (see [26] for details):

1. calculating the pre-envelope of the accelerogram;
2. calculating the envelope itself from the pre-envelope.

The pre-envelope, $z(t)$, of an acceleration time-history, $a(t)$, is defined as in Eq. 3.

$$z(t) = a(t) + i\hat{a}(t) \quad (3)$$

Here, the pre-envelope is a complex-valued signal that has the original acceleration time-history as its real component and the Hilbert transform of the original time-history as its imaginary component ($i = \sqrt{-1}$ and is the complex unit). The Hilbert transform, $\hat{a}(t)$, of $a(t)$ is given by the operation in Eq. 4.

$$\hat{a}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a(\tau)}{t - \tau} d\tau \quad (4)$$

Given the pre-envelope of the original accelerogram one may compute the envelope by taking the modulus of the complex process. That is, the envelope may be calculated using Eq. 5.

$$E(t) = \sqrt{a^2(t) + \hat{a}^2(t)} \quad (5)$$

The Hilbert transform of the accelerogram, $\hat{a}(t)$, is a signal that has the same amplitude spectrum and frequency content as the original signal but with a consistent 90° phase shift applied to all components [16]. This fact proves to be particularly useful for deriving the general form of the envelope function. From Eq. 5 and the knowledge that the integral of a linear combination of functions is the linear combination of the integrals of the individual functions, one may obtain the following expression:

$$\int_{-\infty}^{\infty} E^2(t) dt = \int_{-\infty}^{\infty} a^2(t) dt + \int_{-\infty}^{\infty} \hat{a}^2(t) dt \quad (6)$$

Now recalling Parseval's theorem that dictates equivalence of power between the time and frequency domains as formally stipulated by Eq. 7,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \quad (7)$$

we may state that the two integrals on the right-hand-side of Eq. 6 are equal (as the Hilbert transform acts only to introduce a phase shift and does not influence the amplitude spectrum) and that Eq. 6 may therefore be rewritten as in Eq. 8.

$$\int_{-\infty}^{\infty} E^2(t)dt = 2 \int_{-\infty}^{\infty} a^2(t)dt \quad (8)$$

Now, recalling Eq. 2 we may arrive at the general expression for the envelope function, $E(t)$, to be used in this study.

$$E(t) = \sqrt{\frac{4gI_a}{\pi}} h(t) \quad (9)$$

The Husid pdf, given by $h(t)$, is assumed to be modelled by the pdf of a lognormal distribution which leads to the final expression for the envelope function:

$$E(t) = \sqrt{\frac{4gI_a}{t\sigma\pi\sqrt{2\pi}}} \exp\left[-\frac{(\ln(t) - \mu)^2}{2\sigma^2}\right] \quad (10)$$

Hence, the Husid Envelope Function (HEF) is completely defined for all time provided that one has estimates of the Arias intensity corresponding to the scenario in question as well as estimates of the shape parameters μ and σ that correspond to the mean and standard deviation of the lognormal distribution [26]. The estimates of the Arias intensity may be made using existing empirical models such as those of Travararou et al. [29] or Stafford et al. [27]. The shape parameters must be related to the seismological characteristics of the scenario through new empirical predictive models. The derivation of such models is the subject of the following section.

3 Prediction Models for the HEF Parameters

The optimal shape parameters for every component were obtained using standard maximum likelihood techniques as implemented in the MATLAB function `lognfit`. The shape parameters for every component were then associated with the metadata for each component which consisted of the most typical seismological parameters that define scenarios for seismic design; namely, the moment magnitude M , the rupture distance R_{rup} , the average shear-wave velocity $V_{S,30}$ and the depth to the top of the rupture Z_{tor} . Nonlinear random-effects regression analyses were then conducted using the `nlme` package of R (<http://www.r-project.org>; [17, 20]) in order to obtain empirical models for the shape parameters.

Numerous functional forms were considered and the relative performance of each candidate was assessed in terms of standard statistical metrics such as the Akaike Information Criterion, the Bayesian Information Criterion, the likelihood

ratio test, and the residual variance. The coefficients of the various candidate models were all checked to ensure that they were statistically significant at the 95% confidence level and that strong correlations among these coefficients did not exist. The regression analysis was conducted on the shape parameters corresponding to the individual ground-motion components [26]. The final functional forms are given in Eqs. 11 and 12 while the coefficients of these models are presented in Table 1 in the Appendix.

$$\mu = b_1 + b_2 \mathbf{M} + (b_3 + b_4 \mathbf{M}) \ln \sqrt{R_{\text{rup}}^2 + b_5^2} + b_6 \ln V_{S,30} + b_7 Z_{\text{tor}} \quad (11)$$

$$\ln \sigma = c_1 + c_2 R_{\text{rup}} + c_3 \ln V_{S,30} + c_4 Z_{\text{tor}} \quad (12)$$

The shape parameter μ is normally distributed marginally while the shape parameter σ is lognormally distributed marginally, as demonstrated by [26].

4 Sensitivity Analysis

Once we have defined the mathematical model for the envelope, which is dependent upon the main seismological parameters, a sensitivity analysis is carried out. The aim is to evaluate how sensitive the envelope is to each of these parameters. In Fig. 4, different envelope functions obtained from a given set of seismological parameters and different soil types are shown. The scenario is a strike-slip magnitude of $\mathbf{M} = 6.5$ and a rupture distance of $R_{\text{rup}} = 30$ km (this is required for the prediction of the Arias intensity using the model of Travararou et al. [29]). Consistent with

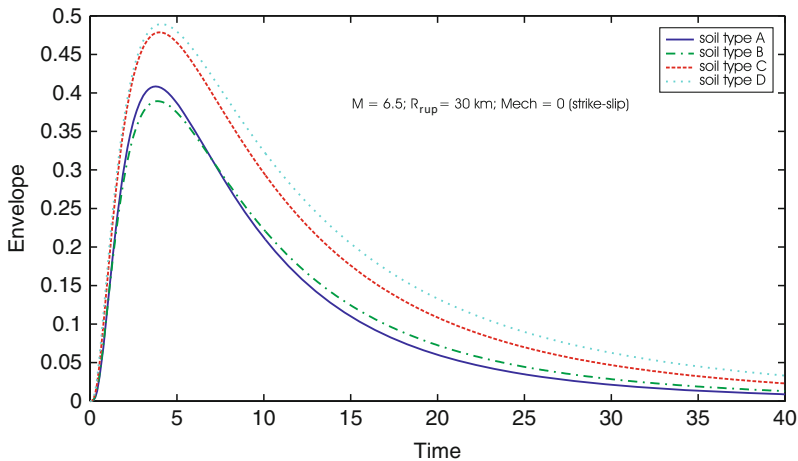


Fig. 4 Comparison between HEFs obtained for a given set of seismological parameters ($\mathbf{M} = 6.5$; $R_{\text{rup}} = 30$ km; $\text{Mech} = 0$) and different NEHRP soil classes

other results in the literature, the median envelope is quite sensitive to soil conditions, in terms of the manner it influences both the maximum value and the implied duration of the motion. In Fig. 5 the same evaluation is presented for different magnitudes and for two soil types (B and C, according to the soil classification scheme adopted by Travararou et al. [29]), while all other parameters are the same.

Finally, two other sensitivities are considered in Figs. 6 and 7. Figure 6 shows the variation of the envelope function with respect to the rupture distance over the range 10–100 km. It is evident that the envelopes are strongly sensitive to this parameter,

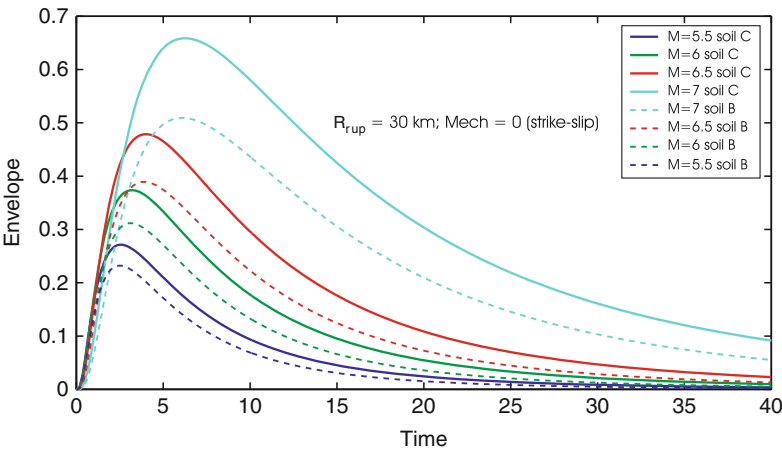


Fig. 5 Comparison between HEFs obtained for a given set of seismological parameters ($R_{rup} = 30$ km; $Mech = 0$) but different magnitudes and for soil classes B and C, using the convention of Travararou et al. [29]

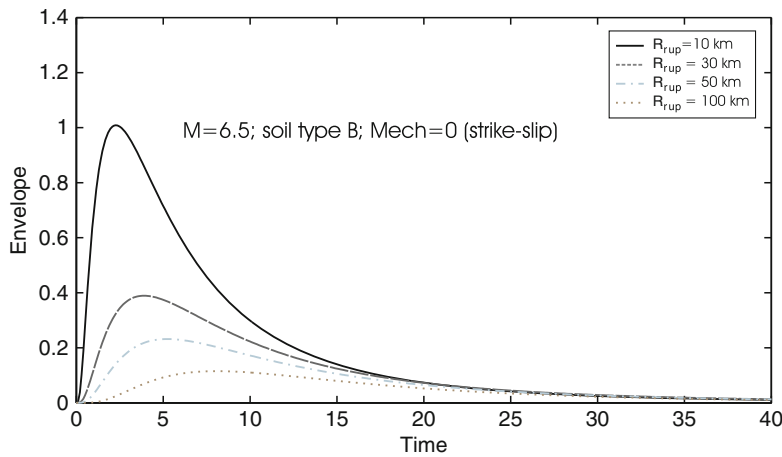


Fig. 6 Comparison between envelope functions obtained for a given set of seismological parameters ($M = 6.5$; NEHRP soil class = B; $Mech = 0$) and different rupture distances

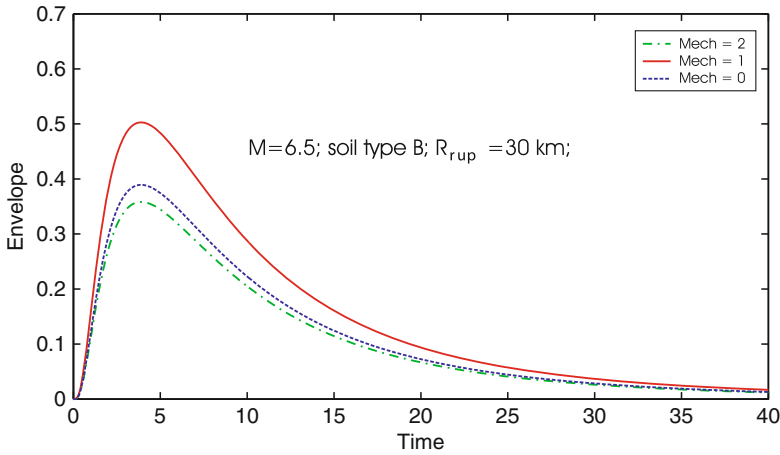


Fig. 7 Comparison between envelope functions obtained for a given set of seismological parameters and different rupture mechanisms: Mech 0 (strike-slip), Mech 1 (normal or normal-oblique), Mech 2 (reverse or reverse-oblique)

showing quite a rapid decrease of its peak and an increasing dispersion of energy in time as the distance R_{rup} increases. Figure 7 shows the influence of the style-of-faulting on the envelope. It should be noted that while Eqs. 11 and 12 are not explicitly functions of the style-of-faulting, the model for Arias intensity [29] that is used for these examples is. Therefore, the value of the Arias intensity that is input to the envelope function expression implicitly contains dependence upon the style-of-faulting. Through the use of this particular model quite a different envelope shape is obtained, but implementation of other models, such as that of Stafford et al. [27], will lead to different degrees of influence.

5 Envelope Variance

In addition to the expected shape of the envelope for a given scenario it is also very desirable to obtain an estimate of the variance of the envelope [26]. It is possible to derive an expression for the variance of the envelope given the knowledge that the individual parameters of I_a , μ and σ are marginally lognormally, normally, and lognormally distributed, respectively (see [26] for more details). In order to derive an analytical expression for the standard deviation of the envelope function we first take the logarithm of Eq. 10 which leads to Eq. 13.

$$\ln E(t) = \frac{1}{2} \left[\ln \left(\frac{4g}{t\pi\sqrt{2\pi}} \right) + \ln I_a - \ln \sigma - \frac{(\ln(t) - \mu)^2}{2\sigma^2} \right] \quad (13)$$

From Eq. 13 one may see that the variance of the logarithmic envelope values may be found by propagating the variances associated with the Arias intensity and the shape parameters. Although the correlations among the three parameters of $\ln I_a$, μ and σ are relatively weak [26], we retain these terms for this generic presentation. It has been shown that the variance of the logarithmic envelope amplitudes can be reduced to the following [26]:

$$\sigma_{\ln E(t)}^2 \approx \frac{1}{4} \left[\sigma_{\ln I_a}^2 + K_C \sigma_\mu^2 + (K_C - 1)^2 \sigma_{\ln \sigma}^2 + \frac{\sqrt{K_C}}{\sigma} \rho_{\ln I_a, \mu} \sigma_{\ln I_a} \sigma_\mu + (K_C - 1) \rho_{\ln I_a, \ln \sigma} \sigma_{\ln I_a} \sigma_{\ln \sigma} + (K_C - 1) \frac{\sqrt{K_C}}{\sigma} \rho_{\mu, \ln \sigma} \sigma_\mu \sigma_{\ln \sigma} \right] \quad (14)$$

where,

$$K_C = \frac{(\ln(t) - \mu)^2}{\sigma^2} \quad (15)$$

A comparison between the full analytical solution, derived above under the assumption that the amplitude of envelope values is lognormally distributed, and the results of 10,000 Monte Carlo simulations is presented in Stafford et al. [26] and shows that this approximate solution is accurate and that the assumption of lognormality is robust.

6 Results

The preceding sections have outlined a full working model that enables one to predict the envelope of an acceleration time-series for a given seismological scenario. The implementation of the model presented herein is straightforward as one simply takes the seismological parameters that define the scenario and insert these into an empirical model for Arias intensity, such as that of Travarasou et al. [29], and in the models presented herein for the shape parameters. In this way, one can directly obtain an estimate of the envelope function and its variance (the variance of the logarithm of the envelope) using Eqs. 10 and 14.

For the purpose of demonstrating the impact of the knowledge of the spectral acceleration some examples have been shown in Fig. 8, where the envelope function is estimated and is compared to a recording from the San Fernando earthquake. The real recorded earthquake is reported (black thin line) with the corresponding analytical envelope. The mean envelope is represented by the black continuous line, while the dashed lines are for the mean $\pm$ one standard deviation.

The same comparison is provided in Fig. 9 for a recording of the 1999 Chi-Chi earthquake. The example fits that are presented in Figs. 8 and 9 are fairly typical of the quality of fit that may be obtained through the use of the model presented herein. However, in numerous cases the implementation of the model will lead to envelopes that do not resemble the actual accelerograms. This is to be expected given just how variable the shapes of accelerograms are in reality [26].

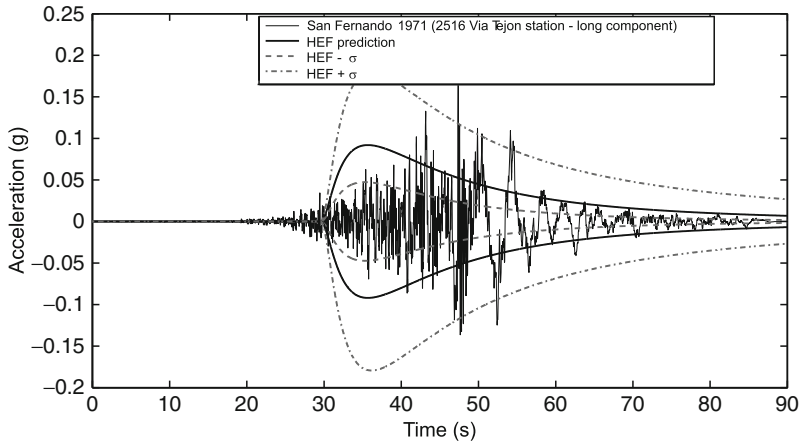


Fig. 8 Example predictions of the HEF for a longitudinal component of ground motion recording of the 1971 San Fernando earthquake 09/02/1971 (2516 Via Tejon station – longitudinal component), $M = 6.61$; $R_{rup} = 55.2$ km; $Mech = 2$ and $V_{S,30} = 280.56$ m/s

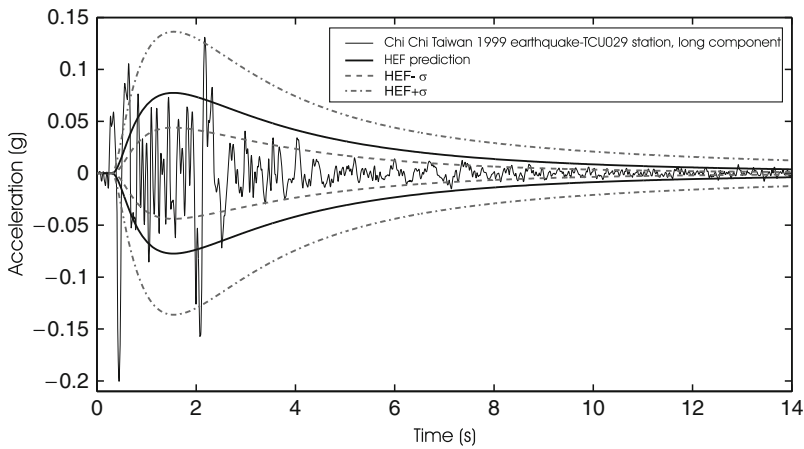


Fig. 9 Example predictions of the HEF for the longitudinal component of ground motion recording of the 1999 Chi-Chi Taiwan earthquake 20/09/1999 (TCU029 station and longitudinal component), $M = 7.62$; $R_{rup} = 28.05$ km; $Mech = 0$; $V_{S,30} = 426.275$ m/s

For the purpose of generating these comparative plots, the envelope functions have been temporally shifted such that their origins coincide with the time at which 0.5% of the total Arias intensity of the real record has been accumulated. This practice is followed purely because each record in the considered dataset has a differing amount and amplitude of noise prior to the actual ground shaking.

7 Prediction of Peak Ground Acceleration

One may appreciate from inspection of Figs. 8 and 9 that although the envelope appears to do a good job of modelling the general shape of the accelerogram, it is not able to encapsulate the peak ground acceleration [26]. Often, these peak acceleration values correspond to rather transient spikes and will not have a significant impact upon structural or geotechnical response. However, there are still numerous applications that make use of estimates of the peak ground acceleration and for this reason it is desirable to provide a mechanism via which peak acceleration values may be related to the envelope function. If one computes the ratio of the peak acceleration and the maximum of the envelope function for all of the components in the strong-motion dataset one finds that these ratios are well-modelled by a lognormal distribution.

The mean value of the peak factor μ_p is very close to 2.5 (the actual value is 2.4934) and the logarithmic standard deviation σ_p is 0.2095, as demonstrated by Stafford et al. [26]. No significant correlation was found by Stafford et al. [26] between the values of the peak factor and the various seismological parameters that are used to derive the envelope function. Therefore, the mean and standard deviation just presented may be used in conjunction with the envelope function to make an estimate of the full distribution of peak ground acceleration values without having to consider any information regarding the particular scenario that is being modelled.

As the peak factor is well-modelled by a lognormal distribution and that ordinates of the envelope function itself are also lognormally distributed, the product of the peak factor p with the maximum value of the envelope function $HEF_{\max} = E(t_{\max})$ provides an estimate of the predicted peak ground acceleration (PGA) that should also be lognormally distributed:

$$PGA = \mu_p E(t_{\max}) \quad (16)$$

Thus, in order to evaluate envelopes consistent with the real PGA , a simple way is to increase the envelope mean by the peak factor – the ratio 2.5 – so that, as shown in Fig. 10, the envelope is simply scaled during the entire phenomenon.

In order to estimate the logarithmic standard deviation of the predicted PGA estimates one may simply take the square root of the sum of the variance of the envelope function (evaluated at the time at which the function is a maximum) and the variance of the peak factor ($\sigma_p^2 = 0.2095^2$), as follows:

$$\sigma_{\ln PGA} = \sqrt{\sigma_p^2 + \sigma_{\ln E(t_{\max})}^2} \quad (17)$$

Because the model used herein to predict the Arias intensity has a logarithmic standard deviation that is a function of the site class, the magnitude of the event and the strength of the Arias intensity itself, no single value for the logarithmic standard deviation of the peak ground acceleration may be provided. A sensitivity analysis of the logarithmic standard deviation of the predicted PGA as a function of these seismological characteristics is provided in Fig. 11 for three different soil classes.

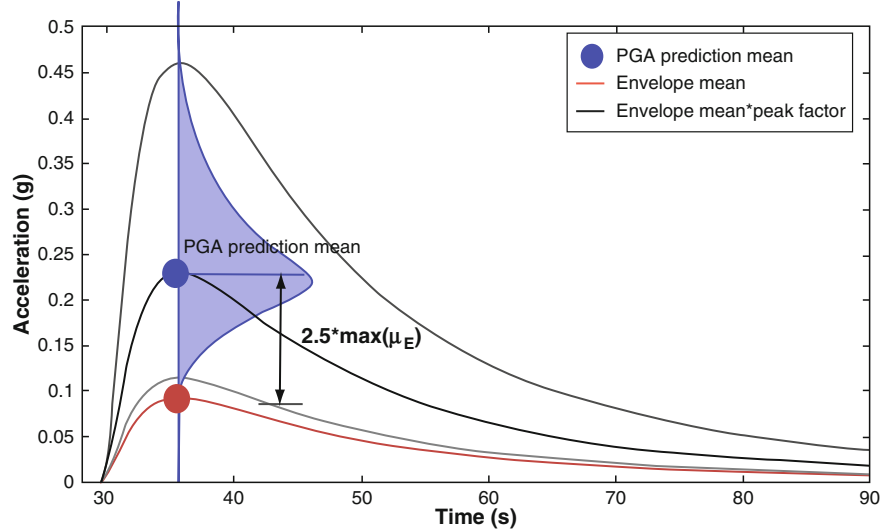


Fig. 10 Mean and distribution of the peak ground acceleration

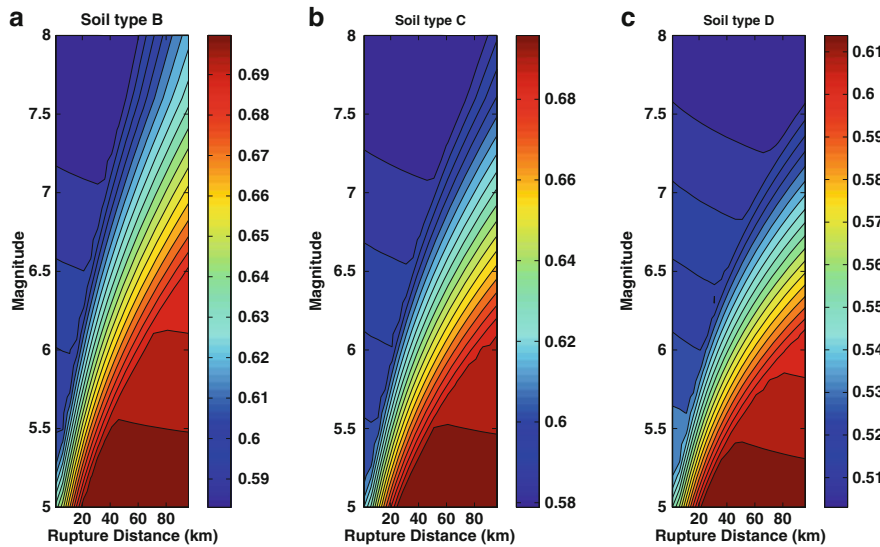


Fig. 11 Dependency of the logarithmic standard deviation of the predicted *PGA* with respect to magnitude and distance for: (a) soil type B; (b) soil type C; and, (c) soil type D. Soil types are consistent with those used by Travararou et al. [29]

As can be noted, the logarithmic standard deviation of the peak ground acceleration varies over the range [0.5,0.7] and generally decreases with increasing magnitude and increases with increasing distance. The variation of these values with respect to

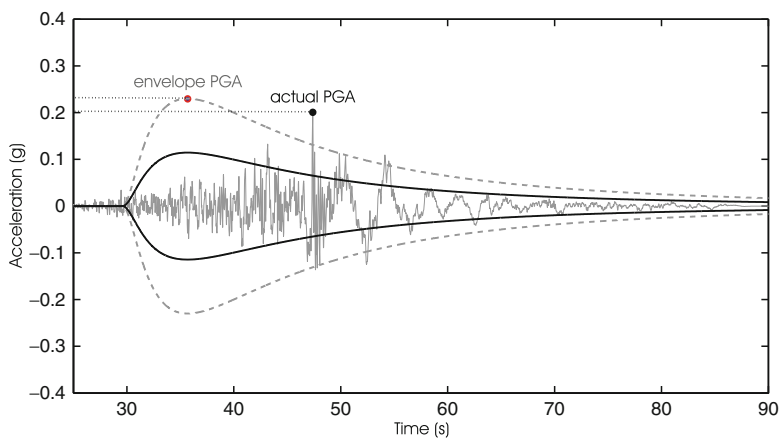


Fig. 12 Comparison between real recorded earthquakes and envelope functions for the longitudinal component of ground motion recording of the 1999 Chi-Chi Taiwan earthquake

the various parameters (magnitude and ground-motion intensity) is strikingly similar to the NGA models, as shown by Stafford et al. [26].

In Fig. 12 an example of the envelope prediction for a scenario is shown by the continuous black line while the estimate of the *PGA* is defined by the grey dashed line. This example demonstrates that while the temporal location of the peak acceleration may not agree with that of the real records in every case, the amplitude of the peak is estimated reasonably well.

8 Summary and Conclusions

This work has addressed the problem of characterizing the time-varying amplitude of strong ground-motion. The primary purpose of computing such an envelope is to enable physically-realistic stochastic accelerograms to be generated. In order to ease the incorporation of this model into routine seismic design, the empirical envelope function has been defined in terms of the main seismological characteristics that define any given earthquake scenario.

The envelope function itself is energy-based in the sense that the overall intensity implied by the envelope is tied directly to the expected Arias intensity corresponding to the scenario for which we are generating the envelope. Furthermore, the shape of the envelope is based directly upon the Husid function that depicts the temporal distribution of energy within an accelerogram. A procedure to obtain the *PGA* from the envelope function has also been presented and has been shown that the prediction is remarkably consistent with the *PGA* predictions of the empirical NGA models.

Appendix

The coefficients for use with Eqs. 11 and 12 are provided below in Table 1.

Table 1 Parameter values and associated 95% confidence intervals (defined as $\pm$ values) for the empirical models for the shape parameters μ (left two columns) and σ (right two columns). The standard deviation terms are defined as follows: σ_E is the inter-event standard deviation, σ_A is the intra-event standard deviation including the component-to-component variability, σ_C is the component-to-component standard deviation, σ_1 is the intra-event standard deviation excluding the component-to-component variability, $\sigma_{T,ARB}$ is the total standard deviation for an arbitrary component, and $\sigma_{T,GM}$ is the total standard deviation corresponding to the geometric mean of two horizontal components

Shape parameter μ		Shape parameter σ	
b_1	-6.0391 ± 0.7769	c_1	0.3151 ± 0.1139
b_2	1.0895 ± 0.1188	c_2	-0.0030 ± 0.0003
b_3	1.8415 ± 0.1316	c_3	-0.0957 ± 0.0165
b_4	-0.2065 ± 0.0193	c_4	-0.0104 ± 0.0096
b_5	5.7575 ± 1.3560		
b_6	-0.1383 ± 0.0217		
b_7	-0.0239 ± 0.0198		
$\sigma_{E,\mu}$	0.3269 ± 0.0491	$\sigma_{E,\ln \sigma}$	0.1774 ± 0.0286
$\sigma_{A,\mu}$	0.2936 ± 0.0059	$\sigma_{A,\ln \sigma}$	0.2244 ± 0.0045
$\sigma_{C,\mu}$	0.1078 ± 0.0030	$\sigma_{C,\ln \sigma}$	0.0821 ± 0.0023
$\sigma_{1,\mu}$	0.2731 ± 0.0065	$\sigma_{1,\ln \sigma}$	0.2089 ± 0.0050
$\sigma_{T,ARB,\mu}$	0.4394 ± 0.0367	$\sigma_{T,ARB,\ln \sigma}$	0.2861 ± 0.0181
$\sigma_{T,GM,\mu}$	0.4259 ± 0.0379	$\sigma_{T,GM,\ln \sigma}$	0.2740 ± 0.0189

References

1. Barnoski, R.L., Maurer, J.R.: Mean-square response of simple mechanical systems to non-stationary random excitation. *J. Appl. Mech. ASME* **36**, 221–227 (1969)

2. Caughey, T.K., Stumpf, H.J.: Transient response of a dynamic system under random excitation. *J. Appl. Mech. ASME* **28**(3), 563–566 (1961)

3. Chiou, B.S.J., Darragh, R., Gregor, N., Silva, W.: NGA project strong-motion database. *Earthq. Spectra*. **24**(1), 23–44 (2008)

4. Dugundji, J.: Envelopes and pre-envelopes of real waveforms. *IRE Trans. Inform. Theor.* **4**, 53–57 (1958)

5. Fan, F.G., Ahmadi, G.: Nonstationary Kanai-Tajimi models for El Centro 1940 and Mexico City 1985 earthquakes. *Probabilist. Eng. Mech.* **5**, 171–181 (1990)

6. Housner, G.W.: Characteristics of Strong-Motion Earthquakes. *Bull. Seism. Soc. Am.* **37**, 19–31 (1947)

7. Iyengar, R.N., Iyengar, K.T.S.: A nonstationary random process model for earthquake accelerograms. *Bull. Seismol. Soc. Am.* **59**, 1163–1188 (1969)

8. Jangid, R.S.: Response of a SDOF system to non-stationary earthquake excitation. *Earthq. Eng. Struct. Dyn.* **33**, 1417–1428 (2004)

9. Jennings, P., Housner, G.W., Tsai, N.: Simulated earthquake motions for design purposes. In: *Proceedings of the 4th World Conference on Earthquake Engineering*, Santiago, Chile, A-1: pp. 145–160 (1968)

10. Kanai, K.: Semi-empirical formula for the seismic characteristics of the ground motion. *Bull. Earthq. Res. Inst. Univ. Tokyo* **35**, 309–325 (1957)
11. Levy, R., Kozin, F., Moorman, R.B.B.: Random processes for earthquake simulation. *J. Eng. Mech. Div-ASCE* **97**, 495–517 (1971)
12. Liang, J., Chaudhuri, S.R., Shinozuka, M.: Simulation of nonstationary stochastic processes by spectral representation. *J. Eng. Mech-ASCE* **133**(6), 616–627 (2007)
13. Lin, Y.K., Yong, Y.: Evolutionary Kanai-Tajimi earthquake models. *J. Eng. Mech. Div-ASCE* **113**(8), 1119–1137 (1987)
14. Liu, S., Jhaveri, D.P.: Spectral simulation and earthquake site properties. *J. Eng. Mech. Div-ASCE* **95**, 1145–1168 (1969)
15. Lutes, L.D., Sarkani, S.: *Stochastic Analysis of Structural and Mechanical Vibrations*. Prentice Hall, Englewood Cliffs (2004)
16. Marple, S.L.: Computing the discrete-time analytic signal via FFT. *IEEE Trans. Signal Proces.* **47**(9), 2600–2603 (1999)
17. Pinheiro, J., Bates, D., DebRoy, S., Sarkar, D., and the R Core team *nlme: linear and nonlinear mixed effects models*. R package version 3.1-89 (2008)
18. Priestley, M.B.: Evolutionary spectra and nonstationary processes. *J. Roy. Stat. Soc. Series B (Methodol)* **27**, 204–237 (1965)
19. Quek, S.T., Teo, Y.P., Balendra, T.: Non-stationary structural response with evolutionary spectra using a seismological input model. *Earthq. Eng. Struct. Dyn.* **19**, 275–288 (1990)
20. R Development Core Team R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. ISBN 3-900051-07-0, URL <http://www.R-project.org> (2008)
21. Rezaeian, S., Der Kiureghian, A.: A stochastic ground motion model with separable temporal and spectral nonstationarities. *Earthq. Eng. Struct. Dyn.* **37**, 1565–1584 (2008)
22. Saragoni, G.R., Hart, G.C.: Simulation of artificial earthquakes. *Earthq. Eng. Struct. Dyn.* **2**, 249–267 (1974)
23. Shinozuka, M., Sato, Y.: Simulation of nonstationary random processes. *J. Eng. Mech. Div-ASCE* **93**, 11–40 (1967)
24. Soong, T.T., Grigoriu, M.: *Random vibration of Mechanical and Structural Systems*. Prentice Hall, Englewood Cliffs (1993)
25. Stafford, P.J., Bommer, J.J.: Empirical equations for the prediction of the equivalent number of cycles of earthquake ground motion. *Soil Dyn. Earthq. Eng.* **29**, 1425–1436 (2009)
26. Stafford, P.J., Sgobba, S., Marano, G.C.: An energy-based envelope function for the stochastic simulation of earthquake accelerograms. *Soil Dyn. Earthq. Eng.* **29**, 1123–1133 (2009)
27. Stafford, P.J., Berrill, J.B., Pettinga, J.R.: New predictive equations for Arias intensity from crustal earthquakes in New Zealand. *J. Seismol.* **13**, 31–52 (2009)
28. Tajimi, H.: A statistical method of determining the maximum response of a building structure during an earthquake. In: *Proceedings of the 2nd World Conference on Earthquake Engineering*, Tokyo and Kyoto II: pp. 781–796 (1960)
29. Travarasrou, T., Bray, J.D., Abrahamson, N.A.: Empirical attenuation relationship for Arias intensity. *Earthq. Eng. Struct. Dyn.* **32**, 1133–1155 (2003)

Sparse Representations in Stochastic Mechanics

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Abstract Computational approaches to systems involving random fields or stochastic processes have to discretise these fields or processes. This produces – when compared to the deterministic case – many variables in the computation, resulting in a very high-dimensional problem. Based on the conviction that the essential stochastic properties of the system are close to some – albeit unknown – lower dimensional manifold, one may try to approximate the response of the system by a data-sparse representation.

The basis for this sparse representation has to be found in the course of the computation. One first approach is to exploit the natural tensor product structure between basis vectors describing the physical/deterministic behaviour and a basis describing the stochastic response. There are two steps involved here: one is to find a good basis for the physical description, and the other to find/compute a good basis for the stochastic part. One well-known example is the Karhunen-Loève expansion, resulting from the eigenvalue analysis of the covariance. One problem is of course that the covariance of the response is not known beforehand. We will discuss on how to approximate the basis along with the solution.

The singular value decomposition, which is very closely related to the Karhunen-Loève expansion, is optimal in that it uses the minimal number of dyadic products. Furthermore, the stochastic part of this product is itself again naturally an element of a tensor product with potentially many factors, containing functions of just one random variable. This fact can be additionally exploited, and also be used to obtain an adaptive approximation of the stochastic part.

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1 Introduction

Computations in stochastic mechanics involve – when compared to the deterministic case – huge amounts of data. This is truly independent of which solution method is used, see [22, 29, 34–36, 45, 46, 50] for an overview. Here we take as a prototypical example a stationary stochastic diffusion problem. Solution methods [36] comprise direct integration, including *Monte Carlo* [10, 41, 47] and its relatives as well as *deterministic* integration methods such as *Smolyak* sparse-grid methods [16, 24, 39, 40, 43, 48], stochastic collocation [6, 37], and stochastic Galerkin methods [2, 4, 5, 17, 21, 28, 30, 33, 44, 53, 54], to name a few of the more popular ones.

In general such a stochastic system will have a solution in a *tensor product* space [33], where one factor describes the physical response, and the other the stochastic variability. In our case the stochastic input are random fields of the diffusion coefficient and the right-hand side, i.e. sinks and sources. Often these are introduced into the computation via their *Karhunen-Loève* expansion, or the equivalent spectral expansion in case of stationary fields. Thus they are already in a natural tensor product format. Theory shows that the solution also lives in a tensor product space, and this is also how the numerical solution ‘ansatz’ looks like: spatial trial functions (usually FEM) with random coefficients. The discrete system to be solved also has a definite tensor product structure for the operator [55]. Although this has been exploited to some limited extent [38] in the solution process, much more is possible.

Here we show how the amount of data may be reduced through a sparsification process. The data – here naturally in a tensor product space – may be approximated by low-rank tensor product. This sparse representation may be kept throughout the computation, thereby not only reducing the amount of data to be handled, but also the computational cost.

In Sect. 2, the model problem will be introduced and theoretical results collected. Sect. 3 introduces the discretisation of random fields, and leads more or less naturally to tensor product representations. After introducing the solution methods in Sect. 4, the structure of the discrete equations will be examined in Sect. 5. On this basis the sparse low rank tensor product approximation will be introduced in Sect. 6 and the computational advantages regarding storage and computational cost shown in Sect. 7, with conclusions and an outlook in Sect. 8.

2 Model Problem

Our model problem is formally one of stationary diffusion. Here it arises from the seepage of groundwater through a porous subsurface rock/sand formation [11, 17, 21, 33, 53].

2.1 Deterministic Problem

We first introduce the deterministic problem, where $\mathcal{G} \subset \mathbb{R}^d$ is the domain occupied by the porous subsurface formation, u is the hydraulic head, κ is the conductivity in Darcy's law, and f and g are sinks and sources, both in the domain and on the Neumann boundary Γ_N with normal vector n . Mass conservation then leads to the following equation:

$$-\nabla \cdot (\kappa(x) \nabla u(x)) = f(x), \quad x \in \mathcal{G} \subset \mathbb{R}^d \quad (1)$$

$$-n \cdot (\kappa(x) \nabla u(x)) = g(x), \quad x \in \Gamma_N \subset \partial \mathcal{G}. \quad (2)$$

On the rest of the boundary $\partial \mathcal{G} \setminus \Gamma_N$ we assume appropriate *Dirichlet* boundary conditions such that the problem is well-posed. Here we will for the sake of simplicity assume them as homogeneous, and also that the conductivity tensor κ is represented by just a scalar field $\kappa(x)$.

It is well-known that a satisfactory theory is possible when the problem Eqs. 1, 2 is reformulated in the following weak or variational form [12, 49]: Find $u \in \mathcal{W}$, such that for all $v \in \mathcal{W}$

$$\begin{aligned} \mathbf{a}(v, u) &:= \int_{\mathcal{G}} \nabla v(x) \cdot (\kappa(x) \nabla u(x)) \, dx \\ &= \int_{\mathcal{G}} v(x) f(x) \, dx + \int_{\partial \Gamma_N} v(x) g(x) \, dS(x) =: \langle f, v \rangle. \end{aligned} \quad (3)$$

Here $\mathcal{W}$ is a *Sobolev-Hilbert* subspace of $H^1(\mathcal{G})$ incorporating the appropriate boundary conditions on the Dirichlet boundary, and S is the surface measure on the Neumann boundary. The continuous bilinear form $\mathbf{a}$ is defined on $\mathcal{W} \times \mathcal{W}$, and the duality pairing $\langle \cdot, \cdot \rangle$ is between $\mathcal{W}$ and its dual space $\mathcal{W}^*$, where f resides.

The conditions introduced so far [12, 49], together with the positivity of κ – which follows from the 2nd law of thermodynamics – and the boundedness of both κ and its inverse in $L_\infty(\Omega)$ guarantee by the *Lax-Milgram* lemma that the problem is well-posed in the sense of *Hadamard*, i.e. existence and uniqueness of a solution, depending continuously on the data $f \in \mathcal{W}^*$ and $\kappa \in L_\infty(\Omega)$.

2.2 Stochastic Problem

In the stochastic model problem, we allow the conductivity κ and the sinks and sources f and g to be random fields defined over some probability space $(\Omega, \mathfrak{A}, \mathbb{P})$, where Ω is the basic probability set of elementary events, $\mathfrak{A}$ a σ -algebra of subsets of Ω , and $\mathbb{P}$ a probability measure. This gives the following equations:

$$-\nabla \cdot (\kappa(x, \omega) \nabla u(x, \omega)) = f(x, \omega), \quad x \in \mathcal{G} \subset \mathbb{R}^d, \quad \omega \in \Omega, \quad (4)$$

$$-n \cdot (\kappa(x, \omega) \nabla u(x, \omega)) = g(x, \omega), \quad x \in \Gamma \subset \partial \mathcal{G}, \quad \omega \in \Omega. \quad (5)$$

Here κ , the stochastic conductivity, is a random field, as well as f and g , the stochastic sinks and sources. Hence also the solution $u(x, \omega)$ will be a random field. These Eqs. 4 and 5 are not very different from their deterministic counterparts Eqs. 1 and 2, except for the requirement that the equality also holds for $\omega \in \Omega$ – technically almost surely (a.s.).

It can be shown [4, 33, 36], that similarly to the deterministic problem of Eq. 3 there is a weak or variational formulation for Eqs. 4 and 5: Find $u \in \mathcal{W} \otimes \mathcal{S}$, such that for all $v \in \mathcal{W} \otimes \mathcal{S}$:

$$\mathbf{a}(v, u) := \mathbb{E}(\mathbf{a}(v(\cdot, \omega), u(\cdot, \omega))) = \mathbb{E}(\langle f(\cdot, \omega), v(\cdot, \omega) \rangle) =: \langle\langle f, v \rangle\rangle, \quad (6)$$

where $\mathbb{E}(b) := \mathbb{E}(b(\omega)) := \int_{\Omega} b(\omega) \mathbb{P}(d\omega)$ is the expected value. Here $\mathcal{S}$ is a Hilbert space of random variables (RVs), in the simplest case we may take $\mathcal{S} = L_2(\Omega, \mathbb{P})$, the space of RVs with finite variance. The continuous bilinear form $\mathbf{a}$ is defined by taking the expected value of Eq. 3, and the duality pairing $\langle\langle \cdot, \cdot \rangle\rangle$ is between the Hilbert tensor product $\mathcal{W} \otimes \mathcal{S}$ and its dual – isomorphic with $\mathcal{W}^* \otimes \mathcal{S}^*$ – where f resides. Although we have used essentially the same symbols in Eq. 6 as in Eq. 3, it will be clear from the context what is meant. In the case $\mathcal{S} = L_2(\Omega, \mathbb{P})$, the solution space is isomorphic to $L_2(\Omega, \mathbb{P}; \mathcal{W})$, the space of $\mathcal{W}$ -valued RVs with finite variance.

Again, the conditions introduced so far [12, 33, 36, 49], together with the boundedness of both κ and its inverse, now in $L_{\infty}(\mathcal{G} \times \Omega)$, guarantee via the Lax-Milgram lemma that the problem is well-posed in the sense of Hadamard, i.e. existence and uniqueness of a solution in $\mathcal{W} \otimes \mathcal{S}$, depending continuously on the data $f \in \mathcal{W}^* \otimes \mathcal{S}^*$ and $\kappa \in L_{\infty}(\mathcal{G} \times \Omega)$.

Apart from the mathematical theoretical grounding, it should be noted that the solution is – almost by definition – an element of a tensor product space. It is this property that we want to exploit subsequently.

3 Discretisation of Random Fields

The random fields entering the Eqs. 4 and 5 or Eq. 6 are in the simplest case described (we take κ as example) by their mean [3, 11, 17, 33, 36]

$$\bar{\kappa}(x) := \mathbb{E}(\kappa_{\omega}(x)) = \mathbb{E}(\kappa(x, \cdot)) \quad \text{and} \quad (7)$$

$$\tilde{\kappa}(x, \omega) := \kappa(x, \omega) - \bar{\kappa}(x), \quad (8)$$

the fluctuating part. Additionally the covariance may be considered at different positions

$$\mathbf{C}_{\kappa}(x_1, x_2) := \mathbb{E}(\tilde{\kappa}(x_1, \cdot) \otimes \tilde{\kappa}(x_2, \cdot)). \quad (9)$$

If the mean $\bar{\kappa}(x) \equiv \bar{\kappa}$ is constant, and $\mathbf{C}_{\kappa}(x_1, x_2) = c_{\kappa}(x_1 - x_2)$ is a function of the separation distance only, the process is called *homogeneous*. Here the representation through the *spectrum* as a *Fourier* sum or integral is well known.

3.1 Karhunen-Loève Expansion

These random fields involve infinitely (actually uncountably) many RVs. Computationally we only can deal with finitely many RVs, so these fields have to be discretised. Most such discretisations come from truncated *product representations*; a very well known one is the Karhunen-Loève expansion (KLE), which in the special case of a homogeneous process is equivalent to the spectral representation [3, 11, 17, 33, 36]. Other names for the Karhunen-Loève expansion are: *proper orthogonal decomposition* (POD), *singular value decomposition* (SVD), *principal component analysis* (PCA).

The KLE is computed from the *Fredholm* eigenvalue problem with the covariance as integral kernel [13], which under suitable conditions defines a compact operator in $L_2(\mathcal{G})$:

$$\int_{\mathcal{G}} \mathbf{C}_{\kappa}(x_1, x_2) g_j(x_2) \, dx_2 = \kappa_j^2 g_j(x_1). \quad (10)$$

As the covariance is symmetric and positive definite, so is the compact operator, hence the non-empty *spectrum* is on the positive real half-axis, with only possible accumulation point the origin. The *eigenvalues* $\{\kappa_j^2\}$ can be ordered by decreasing size, and the *eigenfunctions* $g_j(x)$ are orthogonal. From this follows the Karhunen-Loève representation:

$$\kappa(x, \omega) = \bar{\kappa}(x) + \sum_{j=1}^{\infty} \kappa_j g_j(x) \xi_j(\omega) =: \sum_{j=0}^{\infty} \kappa_j g_j(x) \xi_j(\omega), \quad (11)$$

with *centred*, *uncorrelated*, and *unit variance* RVs $\xi_j(\omega)$, i.e. $\mathbb{E}(\xi_j) = 0$ and $\mathbb{E}(\xi_j \xi_k) = \delta_{j,k}$.

3.2 Singular Value Decomposition

We remind the reader of the singular value decomposition (SVD) of a matrix, written as $\mathbf{W} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T = \sum_j \sigma_j \mathbf{u}_j \mathbf{v}_j^T$, with an orthogonal $\mathbf{U}$ with *orthonormal* columns $\mathbf{u}_j$, similarly with $\mathbf{V}$, and a diagonal $\mathbf{\Sigma} = \text{diag}(\sigma_j)$. The *singular values* σ_j are the square roots of the non-negative eigenvalues of the positive semi-definite matrices $\mathbf{W}^T \mathbf{W}$ or $\mathbf{W} \mathbf{W}^T$. In case the sum is truncated, then this is the best possible approximation to $\mathbf{W}$ with this number of terms in the *Frobenius* norm, the ℓ_2 -norm for matrices. It is also a *low-rank sparse* representation, as we are only using the diagonal elements σ_j , a *one-dimensional* array. In this view, the SVD gives the best possible sparse low-rank approximation to a matrix (a second order tensor).

Now [27] to any random field $w(x, \omega) \in L_2(\mathcal{G}) \otimes L_2(\Omega) \cong L_2(\mathcal{G} \times \Omega) \cong L_2(\Omega; L_2(\mathcal{G}))$ we may associate a linear map $W : L_2(\mathcal{G}) \rightarrow L_2(\Omega)$ via

$$\begin{aligned} W : L_2(\mathcal{G}) \ni v &\mapsto W(v)(\omega) := \langle v(\cdot), w(\cdot, \omega) \rangle_{L_2(\mathcal{G})} \\ &= \int_{\mathcal{G}} v(x) w(x, \omega) \, dx \in L_2(\Omega). \end{aligned} \quad (12)$$

We now look at $C_w := W^* W : L_2(\mathcal{G}) \rightarrow L_2(\mathcal{G})$ when applied in the duality bracket with two arbitrary functions, in order to analyse and recognise its action. This gives for all $u, v \in L_2(\mathcal{G})$:

$$\begin{aligned} \langle u, C_w v \rangle_{L_2(\mathcal{G})} &= \langle u, W^* W v \rangle_{L_2(\mathcal{G})} = \langle W(u), W(v) \rangle_{L_2(\Omega)} \\ &= \mathbb{E} (W(u) W(v)) = \mathbb{E} \left(\int_{\mathcal{G}} u(x) w(x, \omega) \, dx \int_{\mathcal{G}} v(y) w(y, \omega) \, dy \right) \\ &= \int_{\mathcal{G}} \int_{\mathcal{G}} u(x) \mathbb{E} (w(x, \omega) w(y, \omega)) v(y) \, dx \, dy \\ &= \int_{\mathcal{G}} u(x) \int_{\mathcal{G}} C_w(x, y) v(y) \, dy \, dx. \end{aligned} \quad (13)$$

This means that the *covariance operator* C_w is represented by the covariance kernel C_w , and the eigenvalues are identical. Hence the KLE is the singular value decomposition of W [36].

3.3 Low-Rank Approximation

If we *truncate* the KLE in Eq. 11 after the M largest eigenvalues in Eq. 10, where the κ_j are the singular values of W , we obtain an *optimal* – in variance – expansion with just M RVs. This is a perfect example for a low-rank sparse approximation – it uses only M numbers – of a random field [33]. The task in the next sections will be to have something similar for the solution. It can not be the KLE to start with, as the covariance of the solution is unknown before the solution is computed. But we will try to stay as close as possible to the KLE/SVD. But it is not only necessary to have a sparse representation of input and output, but to preserve the sparse format throughout the computation, and to use it to reduce the amount of computation.

4 Solution Methods and Tensorial Character of Solution

We will primarily look at the discretisation of the stochastic part. We assume that the deterministic part Eq. 3, which is inside the expectation in Eq. 6 has been discretised with the reader's favourite method. We use in the following standard finite elements,

but this is of no real consequence, i.e. the spatial part is discretised in space via the finite element ansatz $\{N_n(x)\}$ [12, 49, 56] with the coefficients being RVs:

$$\begin{aligned} u(x, \omega) &= \sum_{n=1}^N u_n(\omega) N_n(x) = [N_1(x), \dots, N_N(x)][u_1(\omega), \dots, u_N(\omega)]^T \\ &= \mathbf{N}^T(x) \mathbf{u}(\omega). \end{aligned} \quad (14)$$

This is a *semi-discretisation* of Eq. 6 and results in [33, 36]:

$$\mathbf{K}(\omega)[\mathbf{u}(\omega)] = \mathbf{f}(\omega). \quad (15)$$

The stochastic solution methods can all be subsumed under the heading of weighted residual or *Petrov-Galerkin* methods. In Eq. 15 we weigh the residuum with the test functions $\{Y_\beta(\omega)\}_{\beta=1, \dots, A}$ and require it to vanish

$$\forall \beta : \quad \mathbb{E}([\mathbf{f}(\omega) - \mathbf{K}(\omega)\mathbf{u}(\omega)]Y_\beta(\omega)) = 0. \quad (16)$$

4.1 Direct Integration

In direct integration methods an ansatz for $\mathbf{u}(\omega)$ is immaterial, and one chooses [33, 36]

$$Y_\beta(\omega) = \delta(\omega - \omega_\beta), \quad (17)$$

this being the *Dirac- δ* at the integration points ω_β giving at each point $\mathbf{u}(\omega_\beta)$. The points ω_β may be chosen randomly as in Monte Carlo (MC), from some sequences as in quasi Monte Carlo (QMC), or from some deterministic integration rule as in Smolyak sparse grid rules.

We collect these collocation data as

$$\mathbf{u}_c = [\dots, \mathbf{u}_\beta, \dots] = (\mathbf{u}_{n,\beta}), \quad (18)$$

and see right away the tensorial structure of the solution; it may be thought of as an element of $\mathbb{R}^N \otimes \mathbb{R}^A$, a discrete representation of $\mathcal{W} \otimes \mathcal{S}$ via the bases $\{N_n(x)\}$ of $\mathcal{W}$ and $\{Y_\beta\}$ of $\mathcal{S}$. If $\mathbb{R}^N \otimes \mathbb{R}^A$ is represented via $\mathbb{R}^{N \times A}$, then $\mathbf{u}_c$ may be thought of as a matrix.

In direct integration methods the data $[\dots, \mathbf{u}_\beta, \dots]$ is usually computed, but not stored; mostly only some functional – rather expected value of some functional of $\mathbf{u}_c$ – is accumulated by adding the appropriate weighted sums.

4.2 Stochastic Collocation

In stochastic collocation [6–8, 37, 52] and stochastic Galerkin methods the second discretisation is performed explicitly by assuming that

$$\mathbf{u}(\omega) = \sum_{\alpha}^A \mathbf{u}_{\alpha} X_{\alpha}(\omega) = \mathbf{u}_X \mathbf{X}(\omega), \quad (19)$$

where the $\{X_{\alpha}(\omega)\}$ are *stochastic* ansatz functions $\mathbf{X}(\omega) = [\dots, X_{\alpha}(\omega), \dots]^T$, e.g. *Hermite* polynomials in *Gaussian* RVs. Similarly to Eq. 18, here $\mathbf{u}_X = [\dots, \mathbf{u}_{\alpha}, \dots] = (u_{n,\alpha})$ is the tensorial object, but now the basis for $\mathcal{S}$ is $\{X_{\alpha}\}$.

Inserting this into Eq. 15 and weighting the residuum with the test functions $\{Y_{\beta}\}$ results in

$$\forall \beta : \quad \mathbb{E}((\mathbf{f}(\omega) - \mathbf{K}(\omega)[\mathbf{u}_X \mathbf{X}(\omega)])Y_{\beta}(\omega)) = 0. \quad (20)$$

Stochastic collocation uses again the test functions of Eq. 17, and has as basic data $\mathbf{u}_c = (u_{n,\beta})$. Staying with this result is sometimes termed as probabilistic collocation. In stochastic collocation the data $\mathbf{u}_c$ is transformed to $\mathbf{u}_X$ in the following way: one chooses the $\{X_{\alpha}\}$ orthonormal in $\mathcal{S}$, i.e. $\mathbb{E}(X_{\alpha}X_{\beta}) = \delta_{\alpha,\beta}$. If the collocation points ω_{γ} are from some integration rule, such that $\sum_{\gamma} w_{\gamma} X_{\alpha}(\omega_{\gamma})X_{\beta}(\omega_{\gamma}) = \delta_{\alpha,\beta}$ with integration weights w_{γ} , we obtain

$$\mathbf{u}_{X,\alpha} = \sum_{\gamma} w_{\gamma} \left(\sum_{\beta} \mathbf{u}_{\alpha} X_{\beta}(\omega_{\gamma}) \right) X_{\alpha}(\omega_{\gamma}) = \sum_{\gamma} w_{\gamma} \mathbf{u}_{c,\gamma} X_{\alpha}(\omega_{\gamma}), \quad (21)$$

and with $\mathbf{X} = (X_{\alpha}(\omega_{\gamma}))_{\alpha,\gamma}$ one has succinctly $\mathbf{u}_X = \mathbf{u}_c \text{diag}(w_{\gamma}) \mathbf{X}^T$.

4.3 Stochastic Galerkin

Stochastic Galerkin (SG) is in fact a *Bubnov-Galerkin* method where the test functions are identical to the ansatz functions $Y_{\beta} \equiv X_{\beta}$ [2, 4, 5, 17, 21, 28, 30, 33, 44, 53, 54]. There is a so-called *non-intrusive* variant of SG, where the weighting in Eq. 20 is performed via numerical integration. Here we want to compute the expectations in Eq. 20 analytically.

In contrast to the other methods, where one solves many systems of the size of the original deterministic problem, the SG requires solution of one huge system. But the difference is that here only integrals of *residua* are needed to compute the Galerkin weighting, whereas in the other methods integrals of the *solution* are needed, making the integrand much more costly to evaluate. On the theoretical side, basic convergence of the Galerkin approximation may be established with *Céa's* lemma [33, 36].

5 Structure of Discrete Equations

We will in the following concentrate on the SG method, as it poses more of a challenge at first sight, since it requires the solution of the huge system Eq. 23.

Introducing the KLE expansion of κ Eq. 11 with M terms into the stiffness matrix in either of Eqs. 15, 16, 20 we obtain:

$$\forall \beta : \quad \mathbb{E} \left(\left(\mathbf{f}(\omega) - \sum_{j=0}^M \xi_j(\omega) \mathbf{K}_j [\mathbf{u}_X(\omega)] \right) X_\beta(\omega) \right) = 0, \quad (22)$$

with $\mathbf{K}_j$ a ‘normal’ stiffness matrix with the ‘material’ $\kappa_j g_j(x)$. Defining matrices $\Delta^{(j)}$ with elements $\Delta^{(j)}_{(\beta,\alpha)} = \mathbb{E}(X_\beta \xi_j X_\alpha)$, that equation may be written as [33, 36]

$$\mathbf{K} \mathbf{u}_X = \left(\sum_{j=0}^M \mathbf{K}_j \otimes \Delta^{(j)} \right) \mathbf{u}_X = \mathbf{f}, \quad (23)$$

where $\mathbf{f} = [\dots, \mathbf{f}_\beta = \mathbb{E}(\mathbf{f}(\omega) X_\beta(\omega)), \dots]$. If $\mathbf{u}_X$ is interpreted as a matrix, the term $\mathbf{K}_j \otimes \Delta^{(j)}$ acts on $\mathbf{u}_X$ like $\mathbf{K}_j \mathbf{u}_X (\Delta^{(j)})^T$.

Now the tensor product structure is exhibited not only for the solution and right-hand side, but also for the operator or matrix. It is worth pointing out, that for SG the matrix $\mathbf{K}$ in Eq. 23 inherits the properties of the operator in Eq. 6, i.e. it is symmetric and positive definite.

6 Sparse Approximations

No matter whether one considers $\mathbf{u}_c$ from Eq. 18 or $\mathbf{u}_X$ from Eq. 19, it represents a huge amount of data. There have been numerous efforts to reduce the amount of data and computation, especially through model reduction and adaptive procedures [1, 9, 14, 15, 19, 28, 38, 51, 55]. In a stochastic sense it is clear that the columns are highly correlated in $\mathbf{u}_c$. It is now the goal to find methods which do not produce such large amounts of data, and just as importantly, also never operate on such large amounts of data, thus allowing more efficient numerical procedures.

6.1 Basic Iteration

The system of Eq. 23 is usually solved via iterative methods [18, 23, 25, 31–33, 36, 42], as they leave the highly structured system matrix untouched and only operate on the k th approximation $\mathbf{u}^k$. Assume some preconditioner $\mathbf{P}$, then a simple iteration would look like

$$\mathbf{u}^{k+1} = \mathbf{u}^k + \mathbf{P}^{-1}(\mathbf{f} - \mathbf{K} \mathbf{u}^k), \quad (24)$$

where of course $\mathbf{P}^{-1}$ means symbolically that a preconditioned system is solved. The fixed point of that iteration is indeed the solution, and the iteration matrix $\mathbf{P}^{-1}\mathbf{K}$ is assumed to have spectral radius less than one, i.e. be a contraction. Practically always the preconditioner also has the tensor product structure of $\mathbf{K}$, e.g. in the simplest case $\mathbf{P} = \mathbf{K}_0 \otimes \Delta^{(0)}$, where $\mathbf{K}_0$ is the deterministic matrix computed with the mean material values $\bar{\kappa}(x)$ and $\Delta^{(0)}_{(\beta,\alpha)} = \mathbb{E}(X_\beta X_\alpha)$.

If no special precautions are taken, the algorithm of Eq. 24 operates on the full matrix $\mathbf{u}_X = [\dots, \mathbf{u}_\alpha, \dots] = (u_{n,\alpha})$, costing $M \times A$ matrix-vector products with the $\mathbf{K}_j$, and $M \times N$ matrix-vector products with the $\Delta^{(j)}$. If $\mathbf{P}$ has the simple structure indicated, then there are additionally A solves with $\mathbf{K}_0$.

6.2 A Sparse Low-Rank Start

If we start with $\mathbf{u}^0 = \mathbf{0}$, the first iteration is just the operation of the preconditioner on the right-hand side $\mathbf{f}$. If the random field $f(x, \omega)$ is also approximated via its R -term Karhunen-Loève expansion, this results in a SVD for the matrix $\mathbf{f} = \sum_r^R \varphi_r \mathbf{f}_r \phi_r^T$ (see [33]).

This has rank R , and the first iteration only costs R solves with $\mathbf{K}_0$ and subsequently $M \times R$ matrix-vector products with the $\mathbf{K}_j$, and $M \times R$ matrix-vector products with the $\Delta^{(j)}$. This can be considerably less than for a full matrix if $R \ll N$ and $R \ll A$, by a factor of $\max(R/N, R/A)$. Also storage for $\mathbf{u}$ is reduced from $N \times A$ to $R \times (N + A)$.

But in the iteration the rank may have gone up by as much as RM (as there are $M + 1$ terms in the sum defining $\mathbf{K}$). So again, if no precautions are taken, we will get close to full rank very quickly.

6.3 SVD on the Fly to Keep It Lean

The idea is to perform or rather *update* the SVD of $\mathbf{u}^k$ so to speak ‘on the fly’ at each iteration, so to have the approximation within some error bound of the full rank tensor. We will symbolically write this rank-reduction operation as $T_\epsilon(\mathbf{u}^k)$. It is a nonlinear operation in terms of $\mathbf{u}$, so the character of the iteration in Eq. 24 changes. It now reads [55]

$$\mathbf{u}^{k+1} = T_\epsilon(\mathbf{u}^k + \mathbf{P}^{-1}(\mathbf{f} - \mathbf{K}\mathbf{u}^k)). \quad (25)$$

The manifolds of low-rank matrices have a more complicated structure than a simple subspace, which is why the iteration in Eq. 25 – which works in practice – is not that easy to analyse theoretically. Nevertheless, there is some general work [20] on such ‘truncated’ approximations, and in our case it shows that the whole process may still be ‘convergent’ – it will not really converge to a fixed point but stagnate in some neighbourhood of the true solution – if the original iteration has a contraction factor less than $1/2$ [20].

7 Implementation of Low-Rank Format Solvers

When implementing a solver that acts on the KL representation of the intermediate solutions, one needs to choose a suitable format to represent those tensor quantities. If $\mathbf{u}_X$ is the quantity of interest given as matrix and $\mathbf{u}_X = \sum_{j=1}^R \mathbf{v}_j \mathbf{w}_j^T = \mathbf{V} \mathbf{W}^T$, with $\mathbf{V} \in \mathbb{R}^{N \times R}$ and $\mathbf{W} \in \mathbb{R}^{A \times R}$, is a decomposition into R rank one tensors, then we choose $\{\mathbf{V}, \mathbf{W}\}$ as representation of the low-rank tensor quantity.

Using this format in linear or nonlinear solvers requires one to adapt the arithmetic therein to that of those tensorial quantities. Namely, addition becomes simply concatenation $\{\mathbf{V}_1, \mathbf{W}_1\} + \{\mathbf{V}_2, \mathbf{W}_2\} = \{[\mathbf{V}_1, \mathbf{V}_2], [\mathbf{W}_1, \mathbf{W}_2]\}$ – a $O(1)$ operation – , multiplication by scalars reduces to multiplication of only one (arbitrarily chosen) component of the quantity $\alpha\{\mathbf{V}, \mathbf{W}\} = \{\alpha\mathbf{V}, \mathbf{W}\}$ and application of operators in tensor product form becomes $(\mathbf{A} \otimes \mathbf{B})\{\mathbf{V}, \mathbf{W}\} = \{\mathbf{A}\mathbf{V}, \mathbf{B}\mathbf{W}\}$. Application of the operator $\mathbf{K}$ on $\{\mathbf{V}, \mathbf{W}\}$ would therefore result in $\{[\mathbf{K}_0\mathbf{V}, \mathbf{K}_1\mathbf{V}, \dots], [\mathbf{\Delta}^{(0)}\mathbf{W}, \mathbf{\Delta}^{(1)}\mathbf{W}, \dots]\}$.

Now the requirement that the representation $\{\mathbf{V}, \mathbf{W}\}$ be equivalent to the KL, i.e. equating $u(x, \omega) = \mathbf{N}(x)^T \mathbf{u}_X \mathbf{X}(\omega) = (\mathbf{V}^T \mathbf{N}(x))^T (\mathbf{W}^T \mathbf{X}(\omega))$ with its Karhunen-Loève expansion $u(x, \omega) = \sum_{j=1}^R \sigma_j u_j(x) \xi_j(\omega)$, we find $\mathbf{V}^T \mathbf{N}(x) = [\dots, \sigma_j u_j(x), \dots]$ and $\mathbf{W}^T \mathbf{X}(\omega) = [\dots, \xi_j(\omega), \dots]$, where we arbitrarily chose to associate the σ_j with the spatial part. Thus from the orthogonality of the KL, we also need $\mathbf{V}^T \mathbf{N}(x)$ and $\mathbf{W}^T \mathbf{X}(\omega)$ to be orthogonal with respect to $L_2(\mathcal{G})$ and $L_2(\Omega)$. This means that

$$\int_{\mathcal{G}} (\mathbf{V}^T \mathbf{N}(x)) (\mathbf{V}^T \mathbf{N}(x))^T dx = \mathbf{V}^T \left(\int_{\mathcal{G}} \mathbf{N}(x) \mathbf{N}(x)^T dx \right) \mathbf{V} = \mathbf{V}^T \mathbf{G}_N \mathbf{V} \quad (26)$$

must be an orthogonal matrix, or, in other words, that $\mathbf{V}$ must be orthogonal with respect to the spatial Gramian $\mathbf{G}_N = \int_{\mathcal{G}} (\mathbf{N}(x) \mathbf{N}(x)^T) dx$. An analogous result holds for $\mathbf{W}$ with respect to the stochastic Gramian $\mathbf{G}_X = \mathbb{E} (\mathbf{X}(\omega) \mathbf{X}(\omega)^T)$. Thus the discrete analog to the KL is the SVD, modified so as to compute orthogonal factors which are orthogonal not with respect to the Euclidian scalar product but to that induced by the Gramians. While this may be difficult to implement efficiently in an SVD algorithm for full matrices, it is particularly easy for the rank reduction operator, which operates on an already low-rank format, as will be seen in the next section.

7.1 Truncation Operator and Strategies

After application of the operator $\mathbf{K}$ in the current iteration, $\mathbf{u}$ is still in low-rank format, however the (numerical) rank of the components $\mathbf{V}$ and $\mathbf{W}$ has greatly

increased. In order to reduce the rank from some R to $R' < R$ we apply the following algorithm:

- Compute the QR decompositions of V , W with respect to the inner products $(\cdot, \cdot)_{G_N}$ and $(\cdot, \cdot)_{G_X}$ by a stabilized, modified Gram-Schmidt procedure:

$$\begin{aligned} (Q_V, R_V) &\leftarrow \text{QR}(V, G_N) \\ (Q_W, R_W) &\leftarrow \text{QR}(W, G_X) \end{aligned}$$

- Compute the SVD of the $R \times R$ matrix $R_V R_W^T$:

$$(\tilde{V}, \Sigma, \tilde{W}) \leftarrow \text{SVD}(R_V R_W^T)$$

- Choose R' according to some truncation criterion, i.e. either fixed or so to achieve a certain approximation accuracy in a given Schatten- p norm with the most common choice the Schatten-2 norm, i.e. the Frobenius norm. Then compute:

$$\begin{aligned} V' &\leftarrow Q_V \tilde{V}_{:,1:R'} \Sigma_{1:R',1:R'} \\ W' &\leftarrow Q_W \tilde{W}_{:,1:R'} \end{aligned}$$

This effectively implements the operator $T_\epsilon : \{V, W\} \mapsto \{V', W'\}$. The complexity of this algorithm can be shown to be $O(R^2(N + A) + R^3)$. The truncation can be applied at different points in the iteration, affecting both efficiency and accuracy of the method. The different possible variants are:

- *After preconditioning*: this corresponds directly to Eq. 25, makes the truncation in the correct norm – namely on that of the solution, not of the residual – and is best amenable to theoretical treatment, however it is very inefficient, since the preconditioner has to be applied to MR vectors.
- *Before preconditioning*: this gives a big performance gain as preconditioning is concerned, since the preconditioner applies now only to R vectors. However, the truncation is now on the residual and the relation between norms may be difficult. The best choice for T_ϵ is now probably to truncate to some relative ϵ .

$$\mathbf{u}^{k+1} = T_\epsilon \left(\mathbf{u}^k + \mathbf{P}^{-1} T_\epsilon (\mathbf{f} - \mathbf{K} \mathbf{u}^k) \right). \quad (27)$$

- *Inside the operator*: the truncation is performed after every addition or subtraction of tensor quantities. This has the advantage that the cubic term in the complexity estimate won't decrease performance too much with this choice. Otherwise, this strategy has the same advantages and disadvantages as 2.

$$\mathbf{u}^{k+1} = T_\epsilon \left(\mathbf{u}^k + \mathbf{P}^{-1} T_\epsilon (\mathbf{f} - \mathbf{K}_\epsilon \mathbf{u}^k) \right). \quad (28)$$

The subscript ϵ in $\mathbf{K}_\epsilon$ here indicates that truncation is performed after every addition.

Most of following analysis will be performed according to the first variant.

7.2 Termination of the Iteration

Iterative solvers usually stop when the absolute or relative residual of the iterate has fallen below some threshold. However, due to the truncation, a termination criterion based on residual only does not necessarily lead to a terminating iteration, since the truncation may be larger than the error or residual allowed by the threshold independent of the conditioning of the discrete equation itself.

An indication of this is that the progress made by the iteration operator is directly reversed by the truncation, leading to stagnating or erratic iterations. Thus, a measure is needed of how much of the updates is actually used in the truncated iteration. Dissecting the truncated iteration formula into single steps gives

$$\Delta \mathbf{v}^n = \Phi_n(\mathbf{u}^n) \quad (29)$$

$$\mathbf{v}^{n+1} = \mathbf{u}^n + \Delta \mathbf{v}^n \quad (30)$$

$$\mathbf{u}^{n+1} = T_\epsilon(\mathbf{v}^{n+1}) \quad (31)$$

$$\Delta \mathbf{u}^n = \mathbf{u}^{n+1} - \mathbf{u}^n \quad (32)$$

where $\mathbf{u}^n$ is the current iterate and Φ_n is the iteration operator, which may depend on the step number n for instationary methods. For the stationary iteration of Eq. 25 we have $\Phi_n = \Phi$ with $\Phi(\mathbf{u}) = \mathbf{P}^{-1}(\mathbf{f} - \mathbf{K}\mathbf{u})$. If the proposed step $\Delta \mathbf{v}^n$ differs sufficiently from the actual step $\Delta \mathbf{u}^n$ we have an indication that either the truncation is too large or, put differently, that the solver is stagnating in a neighbourhood of the solution, and has attained its goal given the prescribed truncation parameter. To have a scalar criterion for this we define the update ratio: $v^n = (\Delta \mathbf{u}^n, \Delta \mathbf{v}^n) / (\Delta \mathbf{v}^n, \Delta \mathbf{v}^n)$, with the scalar product corresponding to the Frobenius norm. For the iteration to proceed efficiently v^n should be very close to 1. If it deviates more than a prescribed threshold, either the truncation parameter should be reduced or the iteration should stop. Thresholds of ± 0.1 to ± 0.2 seem to work good in practice. Note that overshoots ($v > 1$) also indicate stagnation, since the influence of the truncation becomes comparable to that of the iteration operator.

7.3 Convergence and Stagnation

Due to the nature of the modified equations, no true convergence can be expected. However, it is expected that the iterates will stagnate in some neighbourhood of the true solution. In order to control this we want to have some estimate δ_ϵ depending on the truncation parameter ϵ such that for sufficiently large n

$$\|\mathbf{u}^n - \mathbf{u}^*\| \leq \delta_\epsilon, \quad (33)$$

where $\mathbf{u}^*$ is the true solution. We will derive two different estimates on the size of this neighbourhood, depending on the type of solver used and on the assumption on the nature of the perturbation induced by the truncation operator.

For stationary solvers of the form of Eq. 25 we can write the iteration as $\mathbf{u}^{n+1} = \mathbf{M}\mathbf{u}^n + \mathbf{c}$ and get a rough estimate for δ by assuming that the truncation introduces a random perturbation to the sequence of iterates $\mathbf{u}^n$ no larger than ϵ . If the perturbation in step n is $\boldsymbol{\varepsilon}_n$ and the perturbed sequence is $\tilde{\mathbf{u}}^n$, then we have

$$\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^{n+1} = \Phi(\tilde{\mathbf{u}}^n) - \Phi(\mathbf{u}^n) + \boldsymbol{\varepsilon}_n \quad (34)$$

Setting $\mathbf{e}^n = \tilde{\mathbf{u}}^n - \mathbf{u}^n$ leads to

$$\mathbf{e}^n = \mathbf{M}^n \mathbf{e}^0 + \sum_{j=0}^{n-1} \mathbf{M}^{n-1-j} \boldsymbol{\varepsilon}_j \quad (35)$$

and thus

$$\|\mathbf{e}^n\| \leq \|\mathbf{M}^n \mathbf{e}^0\| + \sum_{j=0}^{n-1} \|\mathbf{M}^{n-1-j} \boldsymbol{\varepsilon}_j\| \leq \rho^n \|\mathbf{e}^0\| + \frac{\epsilon}{1-\rho}, \quad (36)$$

where ρ is the spectral radius of $\mathbf{M}$. Since the first term is supposed to go to zero for a convergent method, we expect that the solution will stagnate in some neighbourhood of radius $\delta = \frac{\epsilon}{1-\rho}$ around the true solution $\mathbf{u}^*$.

Another approach which allows tighter estimates under certain conditions is based on [20], where the effect of truncation on iterative methods for the computation of matrix functions is investigated. Suppose that the sequence of iterates $\mathbf{u}^n$ generated by the iteration operator Φ_n converges q -superlinearly to $\mathbf{u}^*$. Let the truncated sequence start with the same (truncated) initial value $\mathbf{v}^0 = T_\epsilon \mathbf{u}^0$ and let $\mathbf{v}^{n+1} = T_\epsilon \Phi_n(\mathbf{v}^n)$. Then, if the true solution $\mathbf{u}^*$ is in the image of the truncation operator, i.e. $\mathbf{u}^* \in \text{im } T_\epsilon$, and further the effect of T_ϵ can be bounded by $\|\mathbf{u} - T_\epsilon \mathbf{u}\| \leq c_\epsilon \|\mathbf{u} - \mathbf{u}^*\|$ then the truncated sequence will also converge $\mathbf{v}^n \rightarrow \mathbf{v}^* = \mathbf{u}^*$, q -superlinearly with the same q -order. In the more likely case that $\mathbf{u}^*$ is not in the image of T_ϵ , i.e. $\mathbf{u}^* \notin \text{im } T_\epsilon$, and $\|\mathbf{u}^* - T_\epsilon \mathbf{u}^*\| \leq c_*$ (plus some more conditions) then for n sufficiently large $\mathbf{v}^n \in B_\delta(\mathbf{u}^*)$ with $\delta = 2c_*$. For methods like conjugate gradients (CG), it can be shown that they behave like certain BFGS methods [26], and can thus be viewed as a q -superlinearly convergent method. However, details still have to be worked out.

7.4 Numerical Examples

As a test problem we solved the 1D diffusion equation in $[0, 1]$ with Dirichlet boundary conditions using the finite element method with 50 linear elements (i.e. $N = 49$ free nodes) for the spatial basis and Hermite chaos in $m = 6$ Gaussians with complete polynomials up to order $p = 4$, giving $A = 10!/6!4! = 210$ stochastic basis functions. The full tensor product basis thus contains $N \cdot A = 10,290$ ansatz

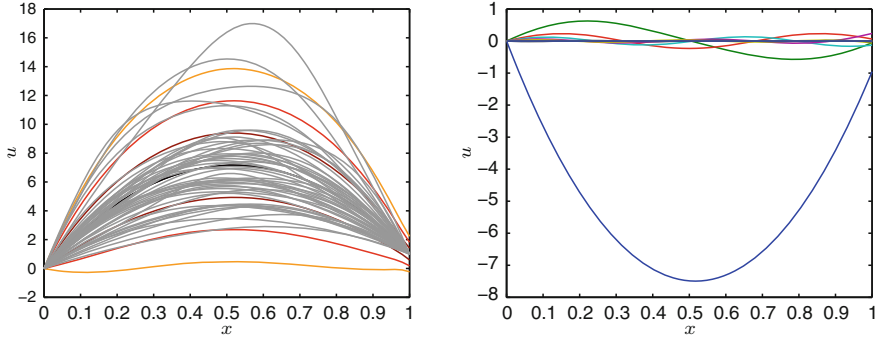


Fig. 1 *Left*: solution u with the sample realizations in grey, mean in black plus one, two and three up times standard deviation. *Right*: the first 10 KL modes of the solution

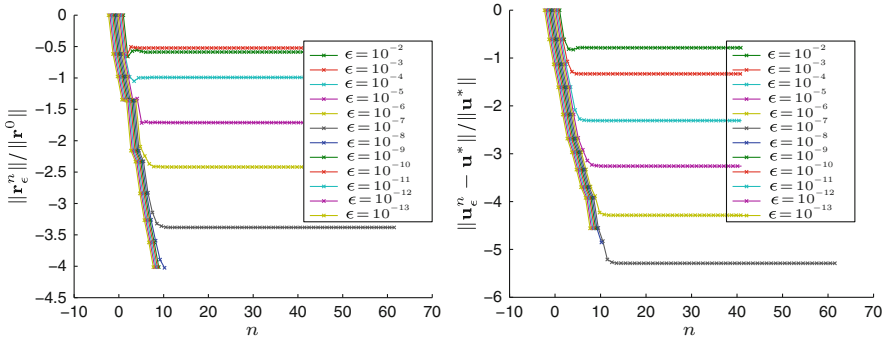


Fig. 2 Relative residual (*left*) and relative error (*right*) for a target relative precision of 10^{-4} for different truncation parameters from 10^{-2} to 10^{-13} . (Iteration numbers were pulled apart a bit to make otherwise overlaying curves better distinguishable.)

functions. Conductivity κ and right hand side f were both taken to be Beta(4, 2) distributed. The solution and the dominant KL modes are shown in Fig. 1.

The progression of relative errors and relative residuals for the above system using a modified CG solver can be seen in Fig. 2. Relative errors were computed with respect to the solution of the full tensor product equation, whose size is small enough to be solved by some direct solver. Clearly visible is the stagnation of the low-rank solver in a neighbourhood of the solution. For small enough truncation parameters $\epsilon < 10^{-7}$ the iterates show no difference to those produced by a full tensor product solver.

The switch to a low-rank solver of course only pays off when the total cost as product of the number of iterations times the cost per iteration is significantly lowered. Figure 3 (*left*) shows that for a sufficiently accurate representation of the solution only approx. 21 modes are required, reducing the required storage by 90%

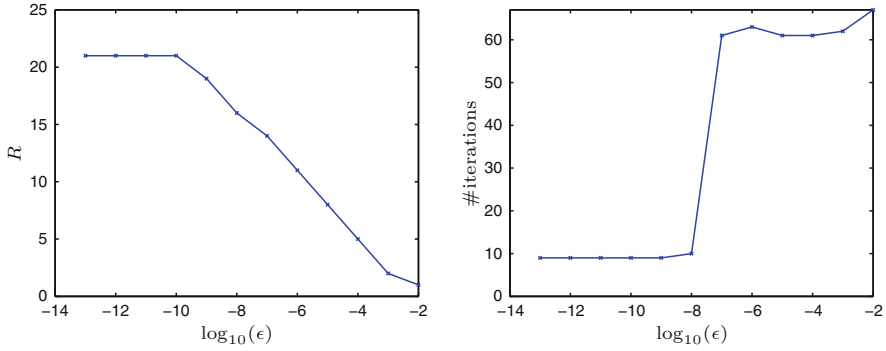


Fig. 3 Numerical rank (*left*) and number of iterations (*right*) for the modified CG solver for different truncation parameters

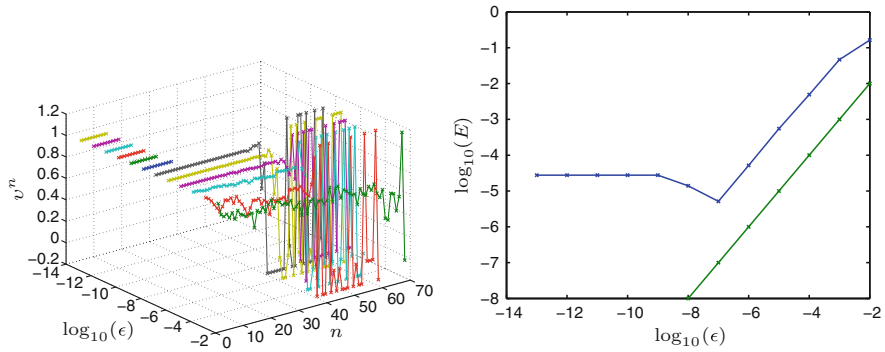


Fig. 4 *Left*: update ratio for different truncation parameters plotted over the iteration number. *Right*: Relative error E (*upper line*) over truncation parameter ϵ . In the *lower line* is again the truncation parameter for comparison

and by 50–80% as intermediate storage is concerned. The number of preconditioner applications per iteration is reduced by 90%. The number of iterations does not change significantly as long as the truncation parameter is small enough as to allow the solver to ‘converge’ as can be seen in Fig. 3 (*right*). However, in the range where the large increase in the number of iterations occurs, stagnation could have been detected much earlier, as can be seen by the wildly oscillating update ratios in Fig. 4 (*left*), and corresponding action been taken like aborting the solver or decreasing the truncation parameter.

Figure 4 (*right*) shows the relation between the relative error at the end of iteration and the truncation parameter ϵ . As expected there is a linear relationship between E and ϵ with $E/\epsilon \simeq 50$, which is in good correspondence with the estimate in Eq. 36.

8 Conclusion

We have shown that for stochastic partial differential equations (SPDEs) the tensor product structure of the solution is not only of theoretical value, but is very important for efficient numerical algorithms. All numerical methods inherit this tensor product structure in their solution, and one may find it if one looks out for it.

It starts with the ‘input’ to the SPDE represented in sparse low-rank format. The computational solution algorithms tend to increase the rank, which will make subsequent steps more costly, as the storage and computational cost is directly proportional to the rank. The SVD ‘on the fly’ is one way to achieve the goal to store and only operate on *sparse* data.

References

1. Acharjee, S., Zabarar, N.: A concurrent model reduction approach on spatial and random domains for the solution of stochastic PDEs. *Int. J. Numer. Meth. Eng.* **66**, 1934–1954 (2006)
2. Acharjee, S., Zabarar, N.: A non-intrusive stochastic Galerkin approach for modeling uncertainty propagation in deformation processes. *Comput. Struct.* **85**, 244–254 (2007)
3. Adler, R.J., Taylor, J.E.: *Random Fields and Geometry*. Springer, Berlin 2007
4. Babuška, I., Tempone, R., Zouraris, G.E.: Galerkin finite element approximations of stochastic elliptic partial differential equations. *SIAM J. Numer. Anal.* **42**, 800–825 (2004)
5. Babuška, I., Tempone, R., Zouraris, G.E.: Solving elliptic boundary value problems with uncertain coefficients by the finite element method: the stochastic formulation. *Comp. Meth. Appl. Mech. Eng.* **194**, 1251–1294 (2005)
6. Babuška, I., Nobile, F., Tempone, R.: A stochastic collocation method for elliptic partial differential equations with random input data. *SIAM J. Numer. Anal.* **45**, 1005–1034 (2007)
7. Baroth, J., Bodé, L., Bressollette, Ph., Fogli, M.: SFE method using Hermite polynomials: An approach for solving nonlinear mechanical problems with uncertain parameters. *Comp. Meth. Appl. Mech. Eng.* **195**, 6479–6501 (2006)
8. Baroth, J., Bressollette, Ph., Chauvière, C., Fogli, M.: An efficient SFE method using Lagrange polynomials: Application to nonlinear mechanical problems with uncertain parameters. *Comp. Meth. Appl. Mech. Eng.* **196**, 4419–4429 (2007)
9. Blatman, G., Sudret, B.: Sparse polynomial chaos expansions and adaptive stochastic finite elements using a regression approach. *C. R. Mécanique* **336**, 518–523 (2008)
10. Caflisch, R.E.: Monte Carlo and quasi-Monte Carlo methods. *Acta Numerica* **7**, 1–49 (1998)
11. Christakos, G.: *Random Field Models in Earth Sciences*. Academic Press, San Diego (1992)
12. Ciarlet, P.G.: *The Finite Element Method for Elliptic Problems*. North-Holland, Amsterdam (1978)
13. Courant, R., Hilbert, D.: *Methods of Mathematical Physics*. Wiley, Chichester (1989)
14. Doostan, A., Ghanem, R.G., Red-Horse, J.: Stochastic model reduction for chaos representations. *Comp. Meth. Appl. Mech. Eng.* **196**, 3951–3966 (2007)
15. Frauenfelder, Ph., Schwab, Chr., Todor, R.A.: Finite elements for elliptic problems with stochastic coefficients. *Comp. Meth. Appl. Mech. Eng.* **194**, 205–228 (2005)
16. Gerstner, T., Griebel, M.: Numerical integration using sparse grids. *Numer. Algorithms* **18**, 209–232 (1998)
17. Ghanem, R.: Stochastic finite elements for heterogeneous media with multiple random non-Gaussian properties. *ASCE J. Eng. Mech.* **125**, 24–40 (1999)
18. Ghanem, R., Kruger, R.: Numerical solution of spectral stochastic finite element systems. *Comp. Methods Appl. Mech. Eng.* **129**, 289–303 (1999)

19. Ghanem, R.G., Pellissetti, M.F.: Adaptive data refinement in the spectral stochastic finite element method. *Commun. Numer. Meth. Eng.* **18**, 141–151 (2002)
20. Hackbusch, W., Khoromskij, B.N., Tyrtshnikov, E.N.: Approximate iterations for structured matrices. *Numer. Math.* **109**, 365–383 (2008)
21. Jarda, M., Su, C.-H., Karniadakis, G.E.: Spectral polynomial chaos solutions of the stochastic advection equation. *SIAM J. Sci. Comput.* **17**, 319–338 (2002)
22. Keese, A.: A review of recent developments in the numerical solution of stochastic PDEs (stochastic finite elements). *Informatikbericht 2003-6*, Institute of Scientific Computing, Department of Mathematics and Computer Science, Technische Universität Braunschweig, Brunswick (2003) <http://opus.tu-bs.de/opus/volltexte/2003/504/>
23. Keese, A., Matthies, H.G.: Parallel solution of stochastic PDEs. *Proc. Appl. Math. Mech.* **2**, 485–486 (2003)
24. Keese, A., Matthies, H.G.: Numerical methods and Smolyak quadrature for nonlinear stochastic partial differential equations. *Informatikbericht 2003-5*, Institute of Scientific Computing, Department of Mathematics and Computer Science, Technische Universität Braunschweig, Brunswick (2003) <http://opus.tu-bs.de/opus/volltexte/2003/471/>
25. Keese, A., Matthies, H.G.: Hierarchical parallelisation for the solution of stochastic finite element equations. *Comput. Struct.* **83**, 1033–1047 (2005)
26. Kolda, T.G., O’Leary, D.P., Nazareth, L.: BFGS with update skipping and varying memory. *SIAM J. Optim.* **8**, 1060–1083 (1998)
27. Krée, P., Soize, C.: *Mathematics of Random Phenomena*. Reidel, D. Dordrecht (1986)
28. Le Maître, O.P., Najm, H.N., Ghanem, R.G., Knio, O.M.: Multi-resolution analysis of Wiener-type uncertainty propagation schemes. *J. Comp. Phys.* **197**, 502–531 (2004)
29. Matthies, H.G., Brenner, C.E., Bucher, C.G., Guedes Soares, C.: Uncertainties in probabilistic numerical analysis of structures and solids – stochastic finite elements. *Struct. Saf.* **19**, 283–336 (1997)
30. Matthies, H.G., Bucher, C.G.: Finite elements for stochastic media problems. *Comp. Meth. Appl. Mech. Eng.* **168**, 3–17 (1999)
31. Matthies, H.G., Keese, A.: Multilevel solvers for the analysis of stochastic systems. In: Bathe, K.-J. (ed.) *Computational Fluid and Solid Mechanics – Proceedings of the 1st MIT Conference* pp. 1620–1622. Elsevier, Amsterdam (2001)
32. Matthies, H.G., Keese, A.: Fast solvers for the white noise analysis of stochastic systems. *Proc. Appl. Math. Mech.* **1**, 456–457 (2002)
33. Matthies, H.G., Keese, A.: Galerkin methods for linear and nonlinear elliptic stochastic partial differential equations. *Comp. Meth. Appl. Mech. Eng.* **194**, 1295–1331 (2005)
34. Matthies, H.G.: Computational aspects of probability in non-linear mechanics. In: Ibrahimbegović A., Brank, B. (eds.) *Engineering Structures under Extreme Conditions. Multi-Physics and Multi-scale Computer Models in Non-linear Analysis and Optimal Design of Engineering Structures under Extreme Conditions*. NATO Science Series III: Computer and System Sciences Vol. 194, IOS Press, Amsterdam (2005)
35. Matthies, H.G.: Quantifying uncertainty: modern computational representation of probability and applications. In: Ibrahimbegović, A., Kožar, I. (eds.) *Extreme Man-Made and Natural Hazards in Dynamics of Structures*. In: *Proceedings of the NATO Advanced Research Workshop PST.ARW981641*. ISBN 953-6953-12-9. Sveučilišna knjižica, Rijeka (2006)
36. Matthies, H.G.: Stochastic finite elements: Computational approaches to stochastic partial differential equations. *Z. Angew. Math. Mech. (ZAMM)* **88**, 849–873 (2008)
37. Nobile, F., Tempone, R., Webster, C.G.: Sparse grid stochastic collocation method for elliptic partial differential equations with random input data. *SIAM J. Numer. Anal.* **46**, 2309–2345 (2008)
38. Nouy, A.: A generalized spectral decomposition technique to solve a class of linear stochastic partial differential equations. *Comp. Meth. Appl. Mech. Eng.* **196**, 4521–4537 (2007)
39. Novak, E., Ritter, K.: The curse of dimension and a universal method for numerical integration. In: Nürnberger, G., Schmidt, J.W., Walz, G. (eds.) *Multivariate Approximation and Splines*, ISNM pp. 177–188. Birkhäuser, Basel (1997)

40. Novak, E., Ritter, K.: Simple cubature formulas with high polynomial exactness. *Constr. Approx.* **15**, 499–522 (1999)
41. Papadrakakis, M., Papadopoulos, V.: Robust and efficient methods for stochastic finite element analysis using Monte Carlo simulation. *Comp. Meth. Appl. Mech. Eng.* **134**, 325–340 (1996)
42. Pellissetti, M., Ghanem, R.: Iterative solution of systems of linear equations arising in the context of stochastic finite elements. *Adv. Eng. Softw.* **31**, 607–616 (2000)
43. Petras, K.: Fast calculation of coefficients in the Smolyak algorithm. *Numer. Algorithms* **26**, 93–109 (2001)
44. Roman, L.J., Sarkis, M.: Stochastic Galerkin method for elliptic SPDEs: A white noise approach. *Discrete Continuous Dyn. Syst.–Series B (DCDS-B)* **6**, 941–955 (2006)
45. Schuëller, G.I.: A state-of-the-art report on computational stochastic mechanics. *Prob. Eng. Mech.* **14**, 197–321 (1997)
46. Schuëller, G.I.: Recent developments in structural computational stochastic mechanics. In: Topping, B.H.V. (ed.) *Computational Mechanics for the Twenty-First Century* pp. 281–310. Saxe-Coburg Publications, Edinburgh (2000)
47. Schuëller, G.I., Spanos, P. D. (eds.): *Monte Carlo Simulation*. Balkema, Rotterdam (2001)
48. Smolyak, S.A.: Quadrature and interpolation formulas for tensor products of certain classes of functions. *Sov. Math. Dokl.* **4**, 240–243 (1963)
49. Strang, G., Fix, G.J.: *An Analysis of the Finite Element Method*. Wellesley-Cambridge Press, Wellesley (1988)
50. Sudret, B., Der Kiureghian, A.: Stochastic finite element methods and reliability. A state-of-the-art report. Report UCB/SEMM-2000/08, Department of Civil & Environmental Engineering, University of California, Berkeley (2000)
51. Wan, X., Karniadakis, G.E.: An adaptive multi-element generalized polynomial chaos method for stochastic differential equations. *J. Comp. Phys.* **209**, 617–642 (2005)
52. Xiu, D., Hesthaven, J.S.: High order collocation methods for differential equations with random inputs. *SIAM J. Sci. Comput.* **27**, 1118–1139 (2005)
53. Xiu, D., Karniadakis, G. E.: Modeling uncertainty in steady state diffusion problems via generalized polynomial chaos. *Comp. Meth. Appl. Mech. Eng.* **191**, 4927–4948 (2002)
54. Xu, X.F.: A multiscale stochastic finite element method on elliptic problems involving uncertainties. *Comp. Meth. Appl. Mech. Eng.* **196**, 2723–2736 (2007)
55. Zander, E., Matthies, H.G.: Tensor product methods for stochastic problems. *Proc. Appl. Math. Mech. (PAMM)* **7**, 2040067–2040068 (2008)
56. Zienkiewicz, O.C., Taylor, R.L.: *The Finite Element Method*, 5th edn. Butterworth-Heinemann, Oxford (2000)

Auto-parametric Stability Loss and Post-critical Behaviour of a Three Degrees of Freedom System

Jiří Náprstek and Cyril Fischer

Abstract Slender structures exposed to a strong vertical component of an earthquake excitation are endangered by auto-parametric resonance effect. This non-linear dynamic process in a post-critical regime caused heavy damages or collapses of many towers, bridges and other structures due to earthquake attack in the epicenter area.

If an amplitude of a vertical harmonic excitation in a structure foundation exceeds a certain limit, vertical response at the top of the structure loses its stability and a dominant horizontal response component arises. In subcritical linear regime only semi-trivial solution being represented by the vertical response component is observed, while horizontal components remain trivial. Therefore, vertical and horizontal response components are independent.

In post-critical regime (auto-parametric resonance) the dynamic non-linear interaction of vertical and horizontal response components occurs. This process results in dominant horizontal response components leading to a failure of the structure. The broadband random non-stationary excitation of the seismic type can be particularly dangerous in this connection.

Qualitatively different post-critical response types have been detected when sweeping throughout the frequency interval beyond the stability limit. Deterministic as well as chaotic response types have been observed. In general they can occur in a steady state, quasi-periodic and completely undetermined regimes including a possible energy transflux among degrees of freedom. Post-critical limit cycles are discussed. The processes of transition from semi-trivial to post-critical state are investigated in case of time limited excitation period.

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1 Introduction

Many studies devoted to dynamics of slender structures (towers, masts, chimneys, bridges, etc.) related with earthquake attack have been published. They are dealing predominantly with an influence of horizontal excitation components. On the other hand a strong vertical excitation component especially in the earthquake epicenter area can be decisive. The vertical component of the subsoil movement caused heavy damages or collapses of many high rise structures due to earthquake attack. Very widely used linear approach, however, usually doesn't provide any interesting knowledge in such a case. It reveals that the origin of these problems consists in auto-parametric resonance effects. This non-linear dynamic process in a post-critical regime caused heavy damages or collapses of many towers, bridges and other structures due to earthquake attack.

In subcritical linear regime the vertical and horizontal response components are independent. If no horizontal excitation is taken into account, no horizontal response component is observed. The semi-trivial solution gives a full image of the structure behaviour. If an amplitude of a vertical harmonic excitation in a structure foundation exceeds a certain limit, vertical response loses its dynamic stability and a dominant horizontal response component through the non-linear interaction of both components is generated. Therefore the system has an auto-parametric character.

Auto-parametric systems have been intensively studied for the last three decades. These investigations are motivated by various technical branches and by basic theoretical research in rational mechanics. A theoretical outline dealing with these systems has been presented probably for the first time in [14]. During this time many papers contributing to analytical, numerical as well as experimental aspects of auto-parametric systems have been published mostly by Tondl, Nabergoj and co-authors, e.g. [22, 32–34]. Many other references can be found. Several monographs, e.g. [11] or [13], presenting a comprehensive overview of partial results and methods have appeared. A couple of papers dealing with auto-parametric systems under deterministic and random excitation has been recently published by the authors of this study [23, 24].

Let us consider the three DOF theoretical model outlined in the Fig. 1. The system is Hamiltonian, [2]. To deduce the governing differential system in the form of Lagrange equations the kinetic and potential energies of the moving system are formulated as follows:

$$T = \frac{1}{2}(M + m)\dot{y}^2 + \frac{1}{2}M\rho^2\dot{\varphi}^2 - m\dot{y}(l\dot{\varphi} + \dot{\xi})\sin\varphi + \frac{1}{2}m(l\dot{\varphi} + \dot{\xi})^2 \quad (a)$$

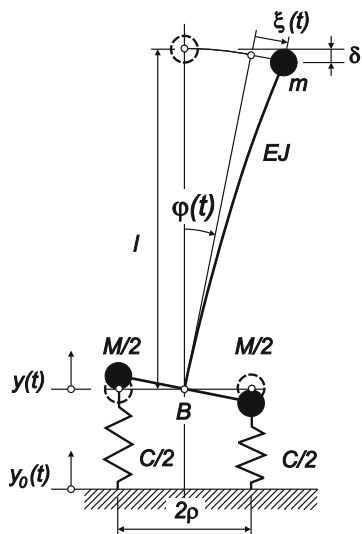
$$U = (M + m)gy - mg[l(1 - \cos\varphi) + \xi\sin\varphi] + \frac{1}{2}C(y - y_0)^2 + \frac{1}{2}Cr^2\varphi^2 + 3EJ/l^3 \cdot \xi^2 \quad (b) \quad (1)$$

$y = y(t)$ – vertical displacement of the B point;

$y_0 = y_0(t)$ – kinematic excitation (seismic random process);

$\varphi = \varphi(t)$ – angular rotation of the system in the B point;

Fig. 1 Outline of the 3-DOF auto-parametric system



$\xi = \xi(t)$ – bending deformation of the vertical console;

M, m – foundation and structure effective masses;

C, EJ – subsoil stiffness, console bending stiffness;

η_c, η_e – viscous components of the C, EJ stiffness (Kelvin);

ϱ, l – geometric parameters.

Non-dimensional response and excitation components

$$\xi_0 = y_0/l, \quad \zeta = y/l, \quad \varphi, \quad \psi = \varphi + \xi/l \quad (2)$$

are useful to be introduced.

When the Lagrange equations are deduced the approximation reflecting an early post-critical state can be adopted:

$$\sin \varphi \approx \varphi; \quad \cos \varphi \approx 1 \quad (3)$$

The governing 3DOF differential system can be formulated as follows:

$$\ddot{\xi} - \kappa_0(\dot{\psi}\varphi) + \omega_0^2(\zeta - \xi_0 + \eta_e(\dot{\zeta} - \dot{\xi}_0)) = 0, \quad (a)$$

$$\ddot{\varphi} - \kappa_1(\dot{\zeta}\varphi) + \kappa_1\dot{\psi} + \kappa_1\dot{\zeta}\dot{\psi} - \kappa_1\omega_2^2\psi + \omega_1^2(\varphi + \eta_c\dot{\varphi}) = 0, \quad (b)$$

$$\ddot{\psi} - (\dot{\zeta}\varphi) + \omega_3^2(\psi - \varphi + \eta_e(\dot{\psi} - \dot{\varphi})) = 0, \quad (c) \quad (4)$$

where following notation has been used:

$$\kappa_0 = \frac{m}{M+m}, \quad \kappa_1 = \frac{m \cdot l^2}{M \cdot \varrho^2},$$

$$\omega_0^2 = \frac{C}{M+m}, \quad \omega_1^2 = \frac{C}{M}, \quad \omega_2^2 = \frac{g}{l}, \quad \omega_3^2 = \frac{6EJ}{m \cdot l^3}. \quad (5)$$

Concerning the excitation process $\zeta_0(t)$, it will be considered as harmonic in the first step. Later the random non-stationary character of $\zeta_0(t)$ will be respected.

2 Semi-trivial Solution and Its Stability

Let us consider the harmonic excitation transformed into the dimensionless form:

$$y_0 = A_0 \sin \omega t \quad \Rightarrow \quad \zeta_0 = a_0 \sin \omega t, \quad A_0 = a_0 \cdot l \quad (6)$$

and assume that the stationary semi-trivial solution exists. Its general form can be written as follows:

$$\zeta_s = a_c \cdot \cos \omega t + a_s \cdot \sin \omega t, \quad \psi = 0, \quad \varphi = 0 \quad (7)$$

Substituting Eq. 7 into the system (4), Eqs. 4b and c are satisfied identically, while Eq. 4a provides the coefficients a_c, a_s doing obvious modifications:

$$a_c = -\frac{a_0 \omega_0^2}{\delta} \omega^3 \eta_c, \quad a_s = \frac{a_0 \omega_0^2}{\delta} (\omega_0^2 - \omega^2 + \omega_0^2 \omega^2 \eta_c^2),$$

$$\delta = (\omega^2 - \omega_0^2)^2 + \omega_0^4 \omega^2 \eta_c^2 \quad (8)$$

Expression (7) together with coefficients (8) represents an approximate simple linear stationary solution of the SDOF system moving in vertical direction being excited kinematically in the point B . The resonance curve of the response amplitude has the form:

$$R_0^2 = a_c^2 + a_s^2 = \frac{a_0^2 \omega_0^4}{\delta} (1 + \omega^2 \eta_c^2) \quad (9)$$

which can be seen in Fig. 2. However the solution being characterized by this curve can be unstable beyond a certain value of the excitation amplitude a_0 in some intervals of the excitation frequency ω . For this reason the stability analysis must be carried out. Very well known general monographs dealing with this topic appeared together with their re-editions, i.e. [8]. Nevertheless, dynamic stability of non-linear systems with one or a couple of degrees of freedom has been discussed using various methods by many authors in problem oriented papers, e.g. [3, 5], or in auto-parametric system focused monographs, e.g. [32].

Let us adopt the linear perturbation approach in order to assess the stability limits of the semi-trivial solution (7). Hence it can be written approximately in the arbitrarily small neighbourhood of the semi-trivial solution:

$$\zeta(t) = \zeta_s(t) + r(t) = \zeta_s(t) + r_c(t) \cos \omega t + r_s(t) \sin \omega t, \quad (a)$$

$$\varphi(t) = 0 + p(t) = p_c(t) \cos \frac{1}{2} \omega t + p_s(t) \sin \frac{1}{2} \omega t, \quad (b)$$

$$\psi(t) = 0 + s(t) = s_c(t) \cos \frac{1}{2} \omega t + s_s(t) \sin \frac{1}{2} \omega t. \quad (c) \quad (10)$$

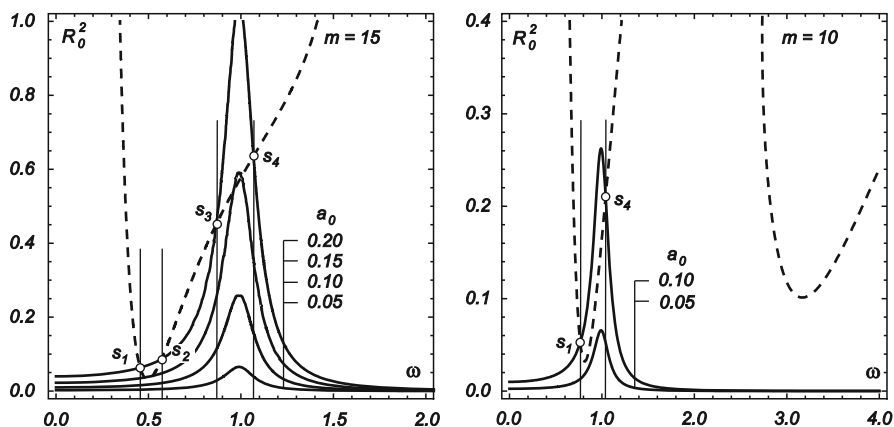


Fig. 2 Relation of resonance curves (solid) and stability limits (dashed);

left picture: $M = 3,990$, $m = 15$, $\eta_c = 0.2$, $\eta_e = 0.2$, $l = 20$, $g = 9.81$, $EJ = 10,000$,

$C = 4,000 > gml/r^2 = 2,943$; $a_0 = 0.2 \Rightarrow s_1 = 0.4548, s_2 = 0.5718, s_3 = 0.8671, s_4 = 1.0680$;

right picture: $m = 10$; $C = 4,000 > gml/r^2 = 1962$; $a_0 = 0.1 \Rightarrow s_1 = 0.7652, s_4 = 1.0396$

The argument (t) will be omitted in further text where possible ($\zeta, \varphi, \psi, r, r_c, r_s, \dots$) etc. Introducing expression (10a) into Eq. 4a and taking into account that ζ_s represents its semi-trivial solution, following equation for perturbation r can be extracted:

$$\ddot{r} + \omega_0^2(r + \eta_c \dot{r}) = 0 \quad (11)$$

Equation 11 is linear and homogeneous. It is obvious that the $\lim_{t \rightarrow \infty} r_c, r_s = 0$ and the stationary solution vanishes. For this reason the vertical response component ζ remains independent and stable in the neighbourhood of the semi-trivial solution ζ_s (on the level of the linear perturbation approach). Let us put now the second column of the expressions (10a–c) into Eqs. 4b, c. Keeping only the linear terms of perturbations p, s and respecting that $r \equiv 0$ one obtains the following differential system:

$$\begin{aligned} \ddot{p} + (\kappa_1 \omega_3^2 \eta_e + \omega_1^2 \eta_c) \dot{p} - \kappa_1 (\omega_3^2 \eta_e - \dot{\zeta}_s) \dot{s} + \\ + (\kappa_1 \omega_3^2 + \omega_1^2) p - \kappa_1 (\omega_2^2 + \omega_3^2) s = 0 \end{aligned} \quad (a)$$

$$\ddot{s} - (\omega_3^2 \eta_e + \dot{\zeta}_s) \dot{p} + \omega_3^2 \eta_e \dot{s} - (\omega_3^2 + \ddot{\zeta}_s) p + \omega_3^2 s = 0 \quad (b) \quad (12)$$

The system (12) is linear as well. However three coefficients include harmonic components due to $\ddot{\zeta}_s, \dot{\zeta}_s$ terms being given by Eq. 7. The system (12) is of the Mathieu type and its solution stability should be verified, cf. for instance [1, 35].

As the next step, the functions p, s in Eqs. 12 should be replaced by means of their first harmonics represented by the third column in Eqs. 10a–c. The method of

harmonic balance enables to obtain the following homogeneous algebraic system for p_c, p_s, s_c, s_s parameters:

$$\begin{bmatrix} -\frac{1}{4}\omega^2 + (\kappa_1\omega_3^2 + \omega_1^2), & \frac{1}{2}(\kappa_1\omega_3^2\eta_e + \omega_1^2\eta_c), \\ -\frac{1}{2}(\kappa_1\omega_3^2\eta_e + \omega_1^2\eta_c), & -\frac{1}{4}\omega^2 + (\kappa_1\omega_3^2 + \omega_1^2), \\ -\omega_3^2 + \frac{1}{4}\omega^2 a_c, & -\frac{1}{2}\omega_3^2\eta_e + \frac{1}{4}\omega^2 a_s, \\ \frac{1}{2}\omega_3^2\eta_e + \frac{1}{4}\omega^2 a_s, & -\omega_3^2 - \frac{1}{4}\omega^2 a_c, \\ -\kappa_1(\omega_2^2 + \omega_3^2) + \kappa_1\frac{1}{4}\omega^2 a_c, & -\frac{1}{2}\kappa_1\omega_3^2\eta_e + \kappa_1\frac{1}{4}\omega^2 a_s \\ \frac{1}{2}\kappa_1\omega_3^2\eta_e + \kappa_1\frac{1}{4}\omega^2 a_s, & -\kappa_1(\omega_2^2 + \omega_3^2) - \kappa_1\frac{1}{4}\omega^2 a_c \\ -\frac{1}{4}\omega^2 + \omega_3^2, & \frac{1}{2}\omega\omega_3^2\eta_e \\ -\frac{1}{2}\omega\omega_3^2\eta_e, & -\frac{1}{4}\omega^2 + \omega_3^2 \end{bmatrix} \begin{bmatrix} p_c \\ p_s \\ s_c \\ s_s \end{bmatrix} = 0 \quad (13)$$

Let us be conscious that Eq. 13 is meaningful only under certain conditions. The system response should be fully or at least nearly stationary in order to be entitled to apply the harmonic balance method. In other words functions p_c, p_s, s_c, s_s should enable to be approximated by constants within the interval of one period or at least to be considered as functions of the “slow time”. Under circumstances of a chaotic or quasi-periodic response with noticeable energy transfer between ζ and φ, ψ components, the harmonic balance method is inapplicable and the system (13) becomes meaningless. Rich references can be addressed to get experiences with early stage of the post-critical processes with dominating chaotic component, see e.g. papers [1, 4, 12], or monographs [17, 25, 29] or even with random character, e.g. [20]. For special considerations regarding non-linear dynamic systems, see [31].

If the above general condition is complied with, p_c, p_s, s_c, s_s can be taken as parameters. The system (13) being homogeneous cannot provide non-trivial solution unless the determinant of its matrix vanishes. The respective determinant can be evolved with respect to R_0^2 amplitude, see Eq. 9, and written in the form of the scalar homogeneous equation:

$$\kappa_1^2\omega^8 R_0^4 - 2\omega^4\kappa_1 A_2(\omega)R_0^2 + A_0(\omega) = 0 \quad (14)$$

where $A_i(\omega)$ are polynomials in ω :

$$\begin{aligned} A_0(\omega) = & \omega^8 + 4\omega^6 \left[(\omega_1^2\eta_c + \kappa_1\omega_3^2\eta_e)^2 - (2\kappa_1 + 1)(1 - \omega_3^2\eta_e^2)\omega_3^2 - 2\omega_1^2 - \omega_3^2 \right] \\ & + 16\omega^4 \left[\omega_1^2\omega_3^2\eta_c(\omega_1^2\eta_c(\omega_3^2\eta_e^2 - 2) + 2\kappa_1\omega_2^2\eta_e) \right. \\ & \quad \left. + 2\gamma\omega_3^2(1 - (\kappa_1 + 1)\omega_3^2\eta_e^2) + ((\kappa_1 + 1)\omega_3^2 + \omega_1^2)^2 \right] \\ & + 64\omega^2 \left[\omega_1^4\omega_3^4\eta_c^2 + \gamma^2\omega_3^4\eta_e^2 - 2\gamma\omega_3^2((\kappa_1 + 1)\omega_3^2 + \omega_1^2) \right] + 256\gamma^2\omega_3^4 \end{aligned}$$

$$A_2(\omega) = \omega^4 - 4\omega^2 \left[\omega_1^2 (1 - \omega_3^2 \eta_c \eta_e) + \omega_3^2 (\kappa_1 (1 - 2\omega_3^2 \eta_e^2) + 1) \right] \\ + 8 \left[\kappa_1 (3\omega_3^4 + (\omega_2^2 + \omega_3^2)^2) + 2\omega_1^2 \omega_3^2 \right]$$

and $\gamma = \omega_1^2 - \kappa_1 \omega_2^2$.

Equation 14 represents a quadratic equation for R_0^2 . Only real positive solution is useful. Negative or complex conjugate roots should be avoided. Conditions for parameters (5) and variable frequency ω could be carried out to fulfil these requests. Consequently $R_0^2 > 0$ as a function of ω consists in a general case either of two or one branches or doesn't exist. Number of real roots of (14) is governed by the behaviour of the discriminant, which has the following form:

$$B_{14}\omega^{14} + B_{12}\omega^{12} + B_{10}\omega^{10} + B_8\omega^8$$

The coefficient B_{14} is always negative, B_8 always positive. Thus, for small ω there exist two real values of R_0^2 . On the other hand, for large ω is the discriminant negative and the semi-trivial solution is stable. According to actual values of the other coefficients, the real values of R_0^2 can occur for one or two intervals of ω .

Two typical examples have been plotted in Fig. 2, (the left picture is a more detailed and a larger scaled grayed item in Fig. 3).

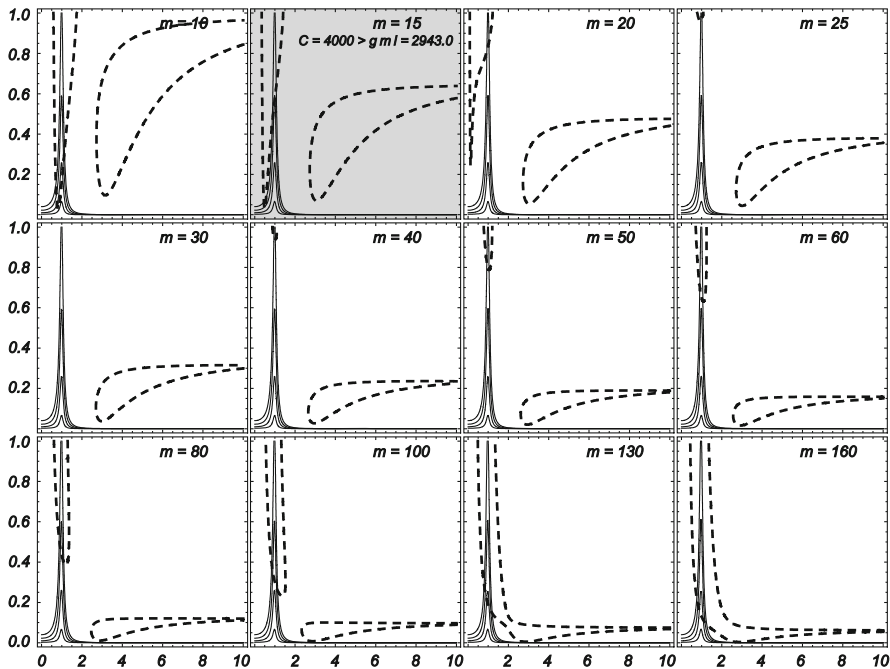


Fig. 3 Resonance curves (solid) and stability limits (dashed) for an increasing value of the mass $m \in (10; 180)$

Resonance curves for a few excitation amplitudes a_0 are plotted using solid lines while the stability limits described by the Eq. 14 are shown in dashed lines for a set of fixed parameters of the system (all parameters used are written in the caption). Points s_1, s_2 and s_3, s_4 in the left picture represent lower and upper limits of intervals where the semi-trivial solution becomes unstable and post-critical response should be investigated. It is obvious that the most sensitive interval where the stability loss can be expected is in the domain of the basic eigen frequency ω_0 as we can see in the left picture – interval $\omega \in (s_3, s_4)$ and also in the right picture – interval $\omega \in (s_1, s_4)$. However the stability limit indicates also another instability interval $\omega \in (s_1, s_2)$ in lower frequency range. This fact is related with a hypothetic “eigen frequency” in the component φ . The right picture demonstrates two separate parts of the particular stability limit.

In order to get a certain overview regarding the influence of basic parameters m (mass of the structure) and C (stiffness of the subsoil) on the stability loss, dynamic response and other properties of the system for various sets of input parameters have been evaluated. Respective results are summarized in Fig. 3 (increasing $m \in (10, 160)$) and in Fig. 4 (increasing $C \in (1,000, 15,000)$). With increasing m a certain drop of system sensitivity in the area of ω_0 against the stability loss is evident $m \in (10, 30)$. The size of the secondary instability area is also decreasing in the same time. Starting at $m = 30$ the resonance domain ω_0 gains importance once again and for $m > 100$ both instability domains join together. Although the cases for $m > 25$ demonstrated in Fig. 3 are rather beyond the static stability and consequently their interpretation can be problematic, this series provides an obvious warning against an existence of an upper limit of m when other parameters remain fixed.

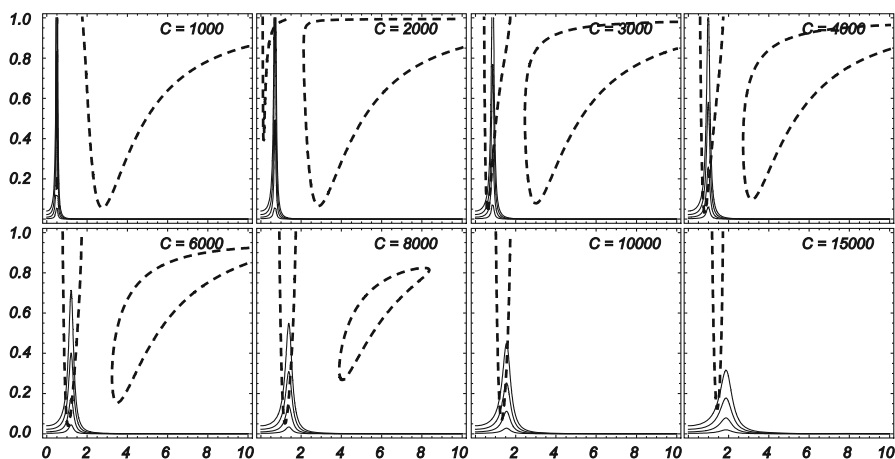


Fig. 4 Resonance curves (solid) and stability limits (dashed) for an increasing value of the spring stiffness $C \in (1,000; 15,000)$

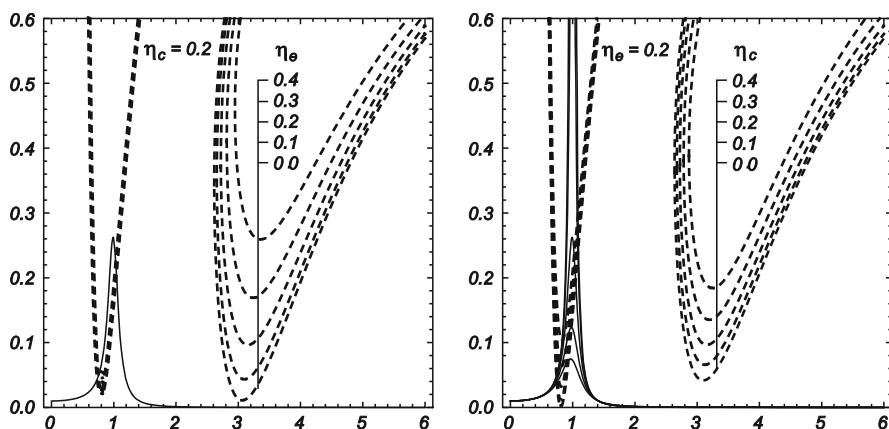


Fig. 5 Influence of the viscous damping on the stability limits (dashed curves); left picture: spring viscosity η_c fixed, viscosity of the vertical deformable console $\eta_e \in (0, 0.4)$ increasing; right picture: $\eta_c \in (0, 0.4)$ increasing, η_e fixed

Just opposite tendency can be noted when increasing the stiffness C . The successive disappear of the secondary instability area with increasing C is obvious. Taking into account the drop of the resonance curve in the resonance domain, one should conclude that increase of C stiffness contributes to the system stabilization.

Let us draw our attention to the influence of the internal damping. Two types of the damping have been taken into account in the basic mathematical model. The first one being quantified by η_c influences strongly the semi-trivial solution and the resonance curve shape, see Fig. 5. The right part demonstrates the system sensitivity to η_c especially in the ω_0 area. Concerning the left part of Fig. 5, we can see that the influence of the viscosity of the vertical console is significant after the stability loss in the post-critical regime. Therefore position and extent of the instability domain is very weakly influenced by the parameter η_e . Concerning the secondary instability area they are not very different regardless whether η_c or η_e is altered within the interval indicated.

3 Post-critical System Response: General Consideration

In order to get overview concerning the system behaviour in frequency intervals of stable semi-trivial and post-critical solutions, several numerical analyses using the governing differential system (4) have been done. The input data correspond with those providing results plotted in Fig. 2 (left picture). Results of simulations are depicted in Fig. 6. In the left column the vertical response component $\zeta(t)$ in dimensionless form, see Eqs. 2, is presented. In the right column the both $\varphi(t)$ and $\psi(t)$ response components are plotted. The both curves almost coincide in this case. The excitation amplitude is always $a_0 = 0.2$. Numerical integrations has been done

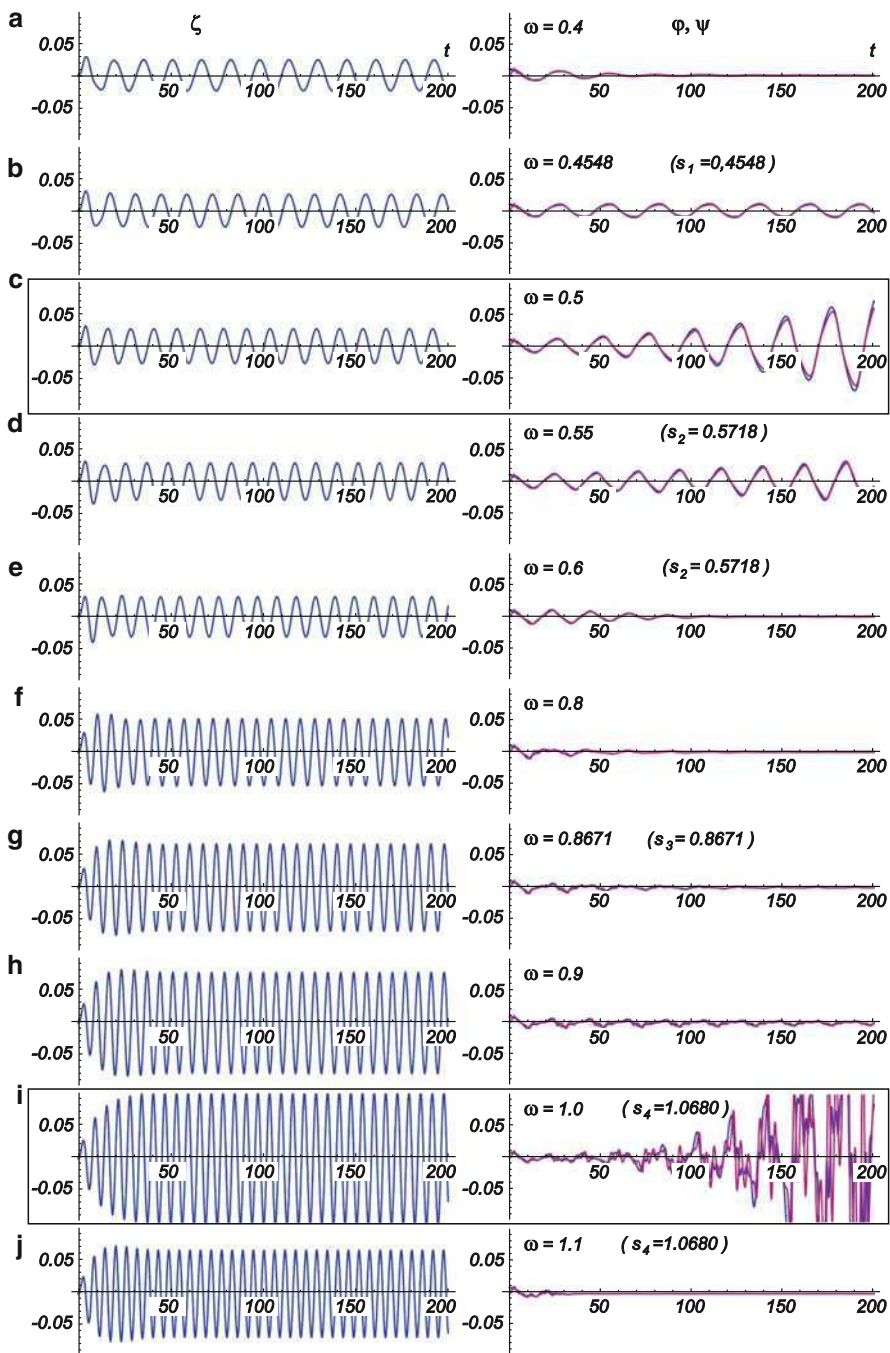


Fig. 6 Time history of the system response; *right column*: $\varphi(t)$, $\psi(t)$ (post-critical response components); *left column*: $\zeta(t)$ (vertical response component at point B); limits of semi-trivial solution stability/instability: $s_1 = 0.4548$, $s_2 = 0.5718$, $s_3 = 0.8671$, $s_4 = 1.0680$

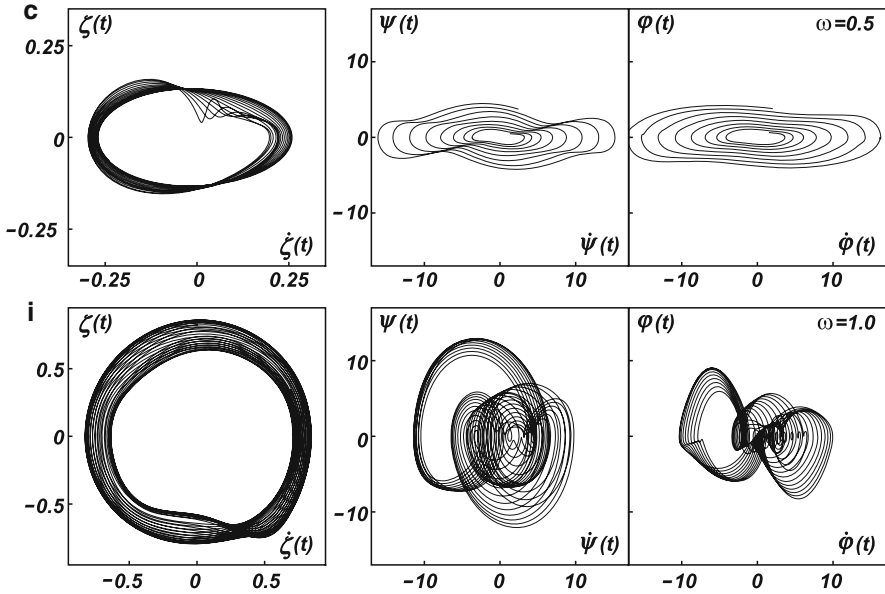
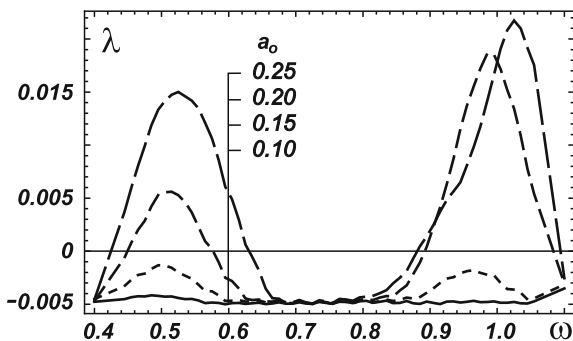


Fig. 7 The phase plot of selected cases of the time history of the system response; *left column*: $\zeta(t)$, *middle column*: $\psi(t)$, *right column*: $\varphi(t)$. Time interval shown here is $t \in (800, 1,000)$. Cf. corresponding rows (c) and (i) in Fig. 6

using the Wolfram-Mathematica code, in particular Gear - predictor-corrector procedure has been repeatedly applied. The stationary state (if any) has been tested after a certain time when the appropriate tests of stationarity passed. The phase plots for selected components of the Fig. 6 are shown in Fig. 7.

The time history of the system response for the excitation frequency $\omega = 0.4$ is demonstrated in the Fig. 6 – row (a). The semi-trivial response is stable despite of a small initial deviation of ψ component. Components $\varphi(t)$, $\psi(t)$ are approaching asymptotically to zero. A transitional state and a beginning of the post-critical state is visible in the row (b) where holds $\omega = s_1$. For details of transition process definition, see e.g. [18]. The noticeable increase of $\varphi(t)$, $\psi(t)$ components within the interval $\omega \in (s_1, s_2)$ is obvious in the rows (c,d), while $\zeta(t)$ still remains on a similar level. Amplitudes of $\varphi(t)$, $\psi(t)$ are approaching to certain horizontal asymptotes for $t \rightarrow \infty$, cf. also Fig. 7. There is a question if such an increase of the horizontal component of the system response is admissible for the structure with respect to various regulations. Overcoming s_2 , the semi-trivial response is re-established and $\varphi(t)$, $\psi(t)$ vanish once again, although vertical component ζ significantly grows up, rows (e-h). Finally in the row (i) very dramatic post-critical system response in the interval $\omega \in (s_3, s_4)$ is obvious, nevertheless $\varphi(t)$, $\psi(t)$ are still approaching to a certain horizontal asymptotes, see Fig. 7. In the row (j) the stable semi-trivial solution without any horizontal components for any arbitrary $\omega > s_4$ arises.

Fig. 8 Values of the largest Lyapunov characteristic exponent λ depending on excitation frequency $\omega \in (0.4, 1.1)$, computed for excitation amplitudes $a_0 = 0.1, 0.15, 0.2, 0.25$



Let us notice that the dynamic stability limits are influenced by the bending stiffness EJ of the vertical console very weakly. However the system response in the post-critical state in particular $\varphi(t)$, $\psi(t)$ components are influenced by this parameter significantly, see Fig. 6, rows (c,i).

The asymptotic behaviour of the response computed for the different excitation frequencies can be assessed also from the point of view of the Lyapunov characteristic exponents (LCE). Numerical procedure follows guidelines published e.g. in [10,28], cf. also the comprehensive review [30]. Figure 8 shows the largest LCE for four excitation amplitudes a_0 depending on the excitation frequency. To relate the LCE with the semi-trivial solution, an obvious almost-zero initial condition has been supposed. Positive values of the LCE indicate the unstable part of the trajectory of the semi-trivial solution. However, even if the LCE are for $\omega \in (s_1, s_2) \cup (s_3, s_4)$ (see Fig. 2) positive, their rather small absolute values permit the trajectories to remain within limits for the whole simulation time.

This preliminary analysis shows that within resonance domains the semi-trivial response can remain in force although post-critical response type is rather typical. Three basic categories of post-critical response can be noticed:

(a) – Response amplitudes are either weakly variable or constant.

In such a case the system response can be studied by means of the harmonic balance method. It eliminates an oscillatory response character investigating (approximately) amplitudes only. Let us introduce the harmonic excitation just like in Eq. 6:

$$y_0 = A_0 \sin \omega t \quad \Rightarrow \quad \zeta_0 = a_0 \sin \omega t, \quad A_0 = a_0 \cdot l \quad (6')$$

In the post-critical state all response components are non-trivial. Therefore expecting a single mode response, following approximate expressions can be written:

$$\zeta(t) = R_c(t) \cos \omega t + R_s(t) \sin \omega t, \quad R^2(t) = R_c^2(t) + R_s^2(t) \quad (a)$$

$$\varphi(t) = P_c(t) \cos \frac{1}{2} \omega t + P_s(t) \sin \frac{1}{2} \omega t, \quad P^2(t) = P_c^2(t) + P_s^2(t) \quad (b)$$

$$\psi(t) = S_c(t) \cos \frac{1}{2} \omega t + S_s(t) \sin \frac{1}{2} \omega t, \quad S^2(t) = S_c^2(t) + S_s^2(t) \quad (c) \quad (15)$$

To make a decision between single and multi harmonic approximation is a delicate question. For some interesting experiences, see [26]. Another opinion can be found in [16], when internal resonance or its proximity should be analysed.

Let us put approximations (3) in the system (4). Going through the harmonic balance procedures a differential system for amplitudes

$$\mathbf{X}(t) = [R_c(t), R_s(t), P_c(t), P_s(t), S_c(t), S_s(t)]^T$$

can be obtained:

$$\mathbf{H}(\mathbf{X}) \frac{d\mathbf{X}}{dt} = \mathbf{K}(\mathbf{X}); \quad \mathbf{H}(\mathbf{X}) \in \mathbb{R}^{6 \times 6}, \quad \mathbf{K}(\mathbf{X}) \in \mathbb{R}^6 \quad (16)$$

The detailed structure of the matrix $\mathbf{H}(\mathbf{X})$ and the vector $\mathbf{K}(\mathbf{X})$ is as follows (for sake of simplicity the argument (t) in $R_c(t)$, $R_s(t)$, ... will be omitted):

$$\mathbf{H}(\mathbf{X}) = \frac{1}{2\omega} \begin{bmatrix} 0 & 2 & 0 & 0 & -\frac{1}{2}\kappa_0 P_s & -\frac{1}{2}\kappa_0 P_c \\ -2 & 0 & 0 & 0 & \frac{1}{2}\kappa_0 P_c & -\frac{1}{2}\kappa_0 P_s \\ \kappa_1 P_s & -\kappa_1 P_c & 0 & 1 & 0 & \kappa_1 \\ \kappa_1 P_c & \kappa_1 P_s & -1 & 0 & -\kappa_1 & 0 \\ P_s & -P_c & 0 & 0 & 0 & 1 \\ P_c & P_s & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\mathbf{K}(\mathbf{X}) = \frac{1}{4} \begin{bmatrix} 4(a_0 \eta_c \omega_0^2 + (\omega^2 - \omega_0^2) R_c - \eta_c \omega \omega_0^2 R_s) + \kappa_0 \omega^2 (P_s S_s - P_c S_c) \\ 4(a_0 \omega_0^2 + (\omega^2 - \omega_0^2) R_s + \eta_c \omega \omega_0^2 R_c) - \kappa_0 \omega^2 (P_s S_c + P_c S_s) \\ (P_c (\omega^2 (1 - \kappa_1 R_c) - 4\omega_1^2) - 2\omega \eta_c P_s \omega_1^2) - \\ -\kappa_1 (((R_c - 1) \omega^2 - 4\omega_2^2) S_c + \omega^2 R_s (P_s + S_s)) \\ (P_s (\omega^2 (1 + \kappa_1 R_c) - 4\omega_1^2) + 2\omega \eta_c P_c \omega_1^2) + \\ +\kappa_1 (((R_c + 1) \omega^2 + 4\omega_2^2) S_s - \omega^2 R_s (P_c + S_c)) \\ (-P_c R_c - P_s R_s + S_c) \omega^2 + 2\omega_3^2 (2(P_c - S_c) + \omega \eta_e (P_s - S_s)) \\ (P_s R_c - P_c R_s + S_s) \omega^2 + 2\omega_3^2 (2(P_s - S_s) - \omega \eta_e (P_c - S_c)) \end{bmatrix}$$

Approximations (3) are not too far from a simple form (7). However, amplitudes in (3) are assumed to be functions of time. The system (16) for amplitudes $\mathbf{X}(t)$ is meaningful if they are functions of a “slow time”, in other words if their changes within one period $T = 2\pi/\omega$ are small or vanishing and individual steps of the

harmonic balance operation are acceptable. In such a case two response regimes corresponding to excitation (6') can be encountered:

- (a1) *Weakly non-stationary response*. The system (16) cannot be solved analytically and numerical procedure must be applied. As a feedback the variability rate of the $\mathbf{X}(t)$ components is to be checked. If some periods are comparable (or even shorter) with T , results should be rejected. In general, shapes of amplitudes $\mathbf{X}(t)$ are deterministic and periodic with possibly small perturbations.
- (a2) *Stationary response (special case of the previous one)*. The time derivative of the amplitudes $\mathbf{X}(t)$ vanishes and therefore the remaining non-linear algebraic system should be solved:

$$\frac{d\mathbf{X}}{dt} \equiv 0 \Rightarrow \mathbf{K}(\mathbf{X}) = 0; \quad \mathbf{K}, \mathbf{X} \text{- constant fields} \quad (17)$$

Both (a1), (a2) can cover limit cycles which are typical for certain types of the post-critical behaviour. System (16) can provide stable as well as unstable limit cycles, see e.g. [6].

(b) – Response amplitudes are strongly variable.

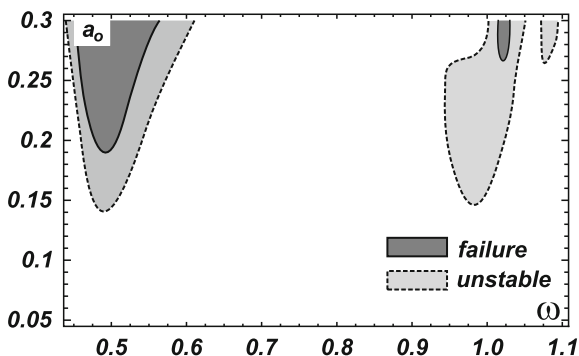
Provided that the response is of any other type than (a), numerical integration of the original system (4) is necessary. The following most important regimes of the response can be observed:

- (b1) *Transition processes*: The systems is asymptotically approaching the stationary state due to sudden energy supply/loss, etc. Response shape is rather deterministic. For some aspects of transition processes and their asymptotic properties, see e.g. [7, 18, 19].
- (b2) *Quasi periodic regime – energy periodic transflux between DOFs*: Beating effects arose due to two or more DOFs interaction. A structure of one period can be complicated. However, the time history repeats and the length of individual periods fluctuates within very limited interval, e.g. [21].
- (b3) *Chaotic regime*: Its existence can be identified by means of Lyapunov exponent testing. For various techniques based on the Lyapunov exponent see e.g. [5, 25, 27].

(c) – Response amplitudes blow up.

This case can be treated as the special case of the previous paragraph (b). Due to the unstable nature of the system under study, excitation which crosses certain limit leads to the collapse of the structure. However, it appears, that the structure can withstand some time limited non-stationary response. Figure 9 can serve as a rough illustration. The amplitude of the response φ is plotted there for various excitation frequencies and amplitudes: white area corresponds to the negligible rocking component φ , gray area shows nonzero but limited amplitude of φ . Return to the stable

Fig. 9 Sensitivity of the system on the frequency and amplitude of the excitation. Light gray area corresponds to the non-zero stationary response in φ and ψ components. Dark gray area exhibits areas of non-stationary (growing) numerical response of the rocking components



state is still possible, if relevant conditions are regained. The dark gray section of the figure corresponds to the cases, where numerical integration fails, i.e. $\varphi \rightarrow \infty$. The final collapse is inevitable. Coincidence of the infected ranges of ω with instability intervals (s_1, s_2) and (s_3, s_4) , see Fig. 2, is evident. Details will be discussed in the Sects. 4 and 5.

Limits between above categories and their regimes are rather smooth due to permanent mixture of deterministic and chaotic/random component of the response. It is obvious that the link of particular cases and their position within domains indicated above depend predominantly on excitation frequency and amplitude. Therefore detailed analysis in subsequent sections will be classified with respect to these parameters.

4 Large Excitation Amplitude

Let us demonstrate the typical system behaviour under excitation with a large amplitude. Resuming the parameter set $C = 10,000$, $EJ = 10,000$, $g = 9.81$, $M = 3,990$, $m = 15$, $\eta_c = 0.2$, $\eta_e = 0.2$, $l = 20$ being used in the section 2, the basic features of the response can be outlined, see Fig. 10. The excitation with the amplitude $a_0 = 0.20$ can be considered as large providing the response amplitude plotted by a bold curve. The system is unstable within intervals $\omega \in (s_1, s_2)$ and $\omega \in (s_3, s_4)$, see hatched (squared) areas in the figure. In order to assess a real behaviour, three types of analysis have been done.

Let us follow respective results in the Fig. 11. The left part concerns amplitudes R of the vertical component $\zeta(t)$, while right part amplitudes S of the dimensional-less horizontal (or angular) component $\psi(t)$. The curves plotted in dashed lines have been obtained using algebraic system (17), thick solid curves have been provided by means of the system (16) and finally thin solid curves represent results of direct simulations on the basis of the differential system (4), cf. adequate sections in [9], or [15].

Fig. 10 Instability intervals of the system under large excitation amplitude ($a_0 = 0.20$), bold smooth curve

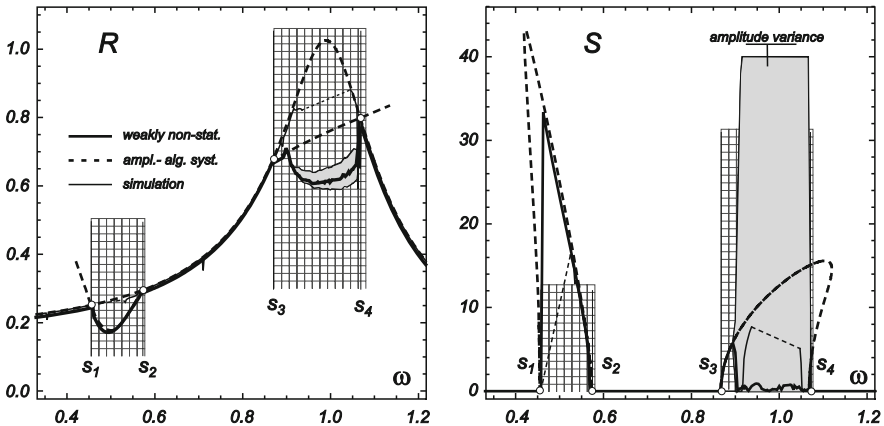
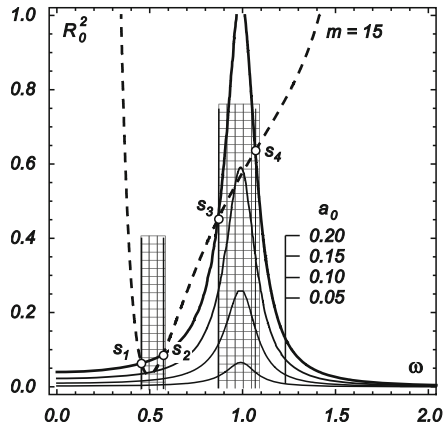


Fig. 11 System response amplitudes evaluated by means of three different techniques; *left*: vertical component $\zeta(t) \rightarrow R$, *right*: horizontal component $\psi(t) \rightarrow S$

As the simulations proved to have considerable chaotic component especially in the interval $\omega \in (s_3, s_4)$, variance of the amplitude has been evaluated and corresponding areas shown as hatched (squared) in Fig. 11. It is obvious that an applicability of results obtained by semi-analytical procedures, Eqs. 16 and 17 is very limited. They can be accepted in the lower interval $\omega \in (s_1, s_2)$, where solution of differential system (16) provides the most realistic results, see also time history in the Fig. 6c and d. The algebraic system leads here to twofold solution and below s_1 unstable branch should be eliminated. However, simulation gives similar results especially in the upper part of the interval.

In the upper interval $\omega \in (s_3, s_4)$ the high degree of instability is visible, cf. Fig. 6h–j. It applies mainly the S component. Obviously analytically obtained static amplitudes (dashed curves) are not realistic and only simulation results providing chaotic results are representative. This is the case discussed in the paragraph (c)

above, see also the corresponding dark parts of Fig. 9. The part of the dashed curve above s_4 is not accessible neither numerically nor experimentally being unstable (only SDOF nonlinear systems can deal with this branch having no possibility to escape into the stable state due to some other degree of freedom).

5 Medium and Small Excitation Amplitudes

The same three DOF system has been investigated for the excitation amplitude $a_0 = 0.15$. Frequency response curve of the semi-trivial solution intersects the dashed stability limit once again, see Fig. 12. There arose two instability intervals once again. However, their width became significantly smaller than in the previous case. This time the post-critical state enables to be investigated using differential or algebraic system (16) or (17) respectively. However some detailed properties need to be investigated using simulation. Observing time history of the response components, see Fig. 13, the transition effects should be analysed.

Inside of the lower instability interval (mean $\omega = 0.5010$) the vertical component doesn't prove any dramatic effects. The only detail to be noticed is a non-significant decrease of the amplitude above the axis only starting approx. $t = 4,000$. This mild de-symmetrizing with respect to time axis is related with a real start of the gradual semi-trivial solution stability loss when horizontal response components are arising. This transition effect represents a slight non-periodical energy transflux between vertical and horizontal DOFs. In other words we can conclude that the stability loss is not any process emerging suddenly, although a velocity of its breaking out can be low or very high remembering a snap-through. Detailed time history is rather simple. For better insight the steady state within the interval $t \in (5,700, 6,000)$ supplements every basic graph on $t \in (0, 6,000)$, see indications in lower parts of

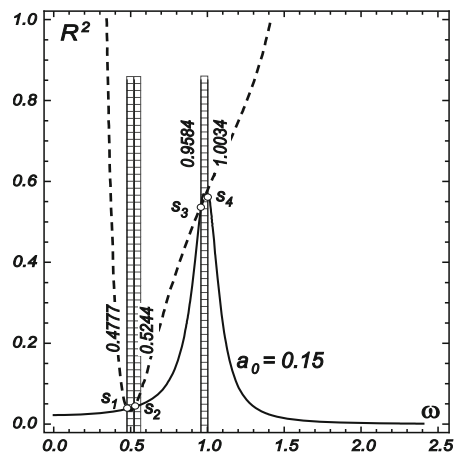


Fig. 12 Instability intervals of the system under medium excitation amplitude ($a_0 = 0.15$)

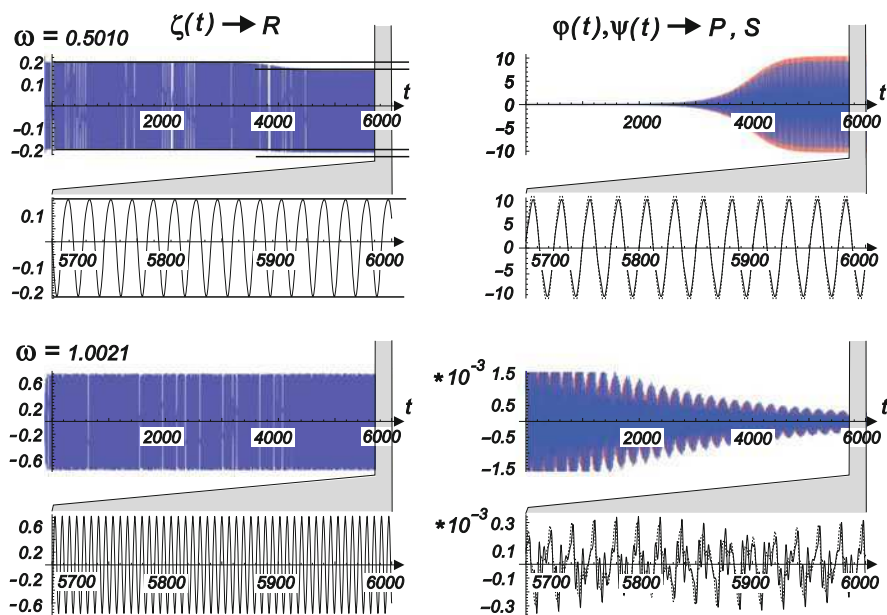


Fig. 13 Time history of the response due to medium excitation amplitude inside of instability intervals; *left column*: vertical component $\zeta(t)$, *right column* horizontal (or rocking) components φ, ψ ; every graph is presented in an original scale $t \in (0, 6,000)$ and magnified scale $t \in (5,700, 6,000)$

the individual pictures on Fig. 13. Concerning the higher instability interval (mean $\omega = 1.0021$), nothing surprising has been detected. Horizontal response components are non-trivial but asymptotically approaching to a small value, making together with the vertical component a stable post-critical state being controlled by non-linear terms. The respective limit cycle which would characterize the post critical response is not elliptic, as it contains multi-harmonic components, see detailed time history in the lower right graph on Fig. 13. The both instability intervals can be seen in the Fig. 9: for the driving amplitude $a_0 = 0.15$ the corresponding frequencies belong to the unstable gray areas but not to the dark failure region.

Let us assess the above results comparing them with Figs. 14 and 15. They demonstrate stable branches of the response amplitude R – Fig. 14 and amplitudes P, S – Fig. 15. The response curve in the left picture of the Fig. 14 proves that within instability intervals only small difference between the original response curve and stable post-critical characteristic exists concerning the vertical component. However, quite large amplitudes of horizontal movement can be achieved, as it follows from Fig. 15. In the same time it can be concluded that no significant differences between results provided using systems (16) – P and (17) – S have arisen. As another evidence regarding these conclusions serve respective variances plotted in right parts of Figs. 14 and 15.

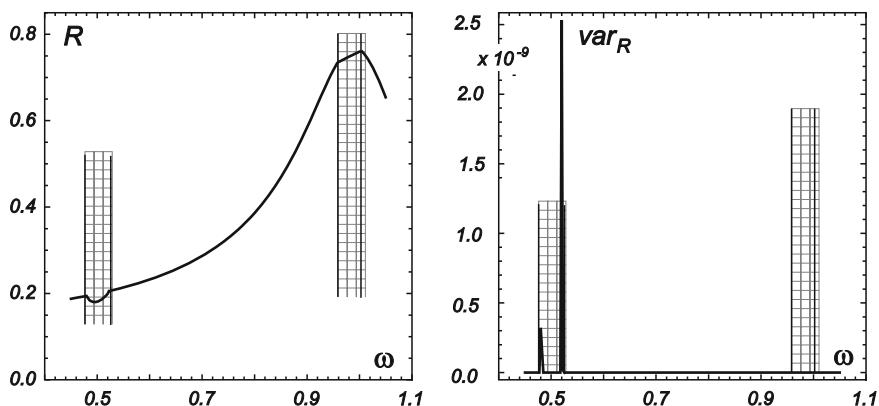


Fig. 14 Vertical response amplitude $\zeta(t) \rightarrow R$ and corresponding variance (*squared area*)

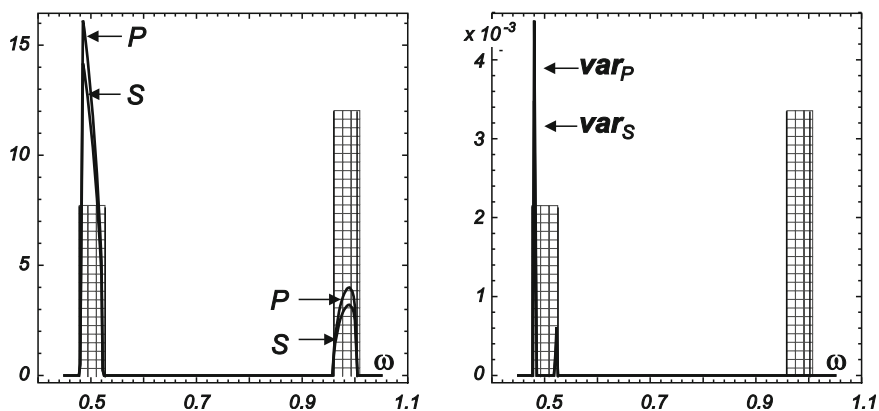


Fig. 15 Horizontal (or rocking) response components $\varphi(t) \rightarrow P, \psi(t) \rightarrow S$ and their variances (*squared area*)

6 Transition from Semi-trivial to Post-critical Response

Let us demonstrate a detailed process of the stability loss and of the transition from the semi-trivial to post-critical response. The underlined parameters of the system examined in previous sections have been slightly increased (subsoil stiffness and internal damping). The actualized parameter set reads now: $C = 16,000$, $EJ = 10,000$, $g = 9.81$, $M = 3,990$, $m = 15$, $\eta_c = 0.6$, $\eta_e = 0.2$, $l = 20$. Outline of instability domains for various amplitudes of excitation and corresponding bifurcation diagrams can be found in Fig. 16. Figure 16a represents resonance curves (*solid*) of the semi-trivial solution for amplitudes $a_0 \in (0.3, 0.6)$. Stability limit (*dashed*) represents the set of bifurcation points delimiting intervals ω where two solutions appear: the unstable semi-trivial solution (*dotted*) and stable post-critical solution (*solid line*).

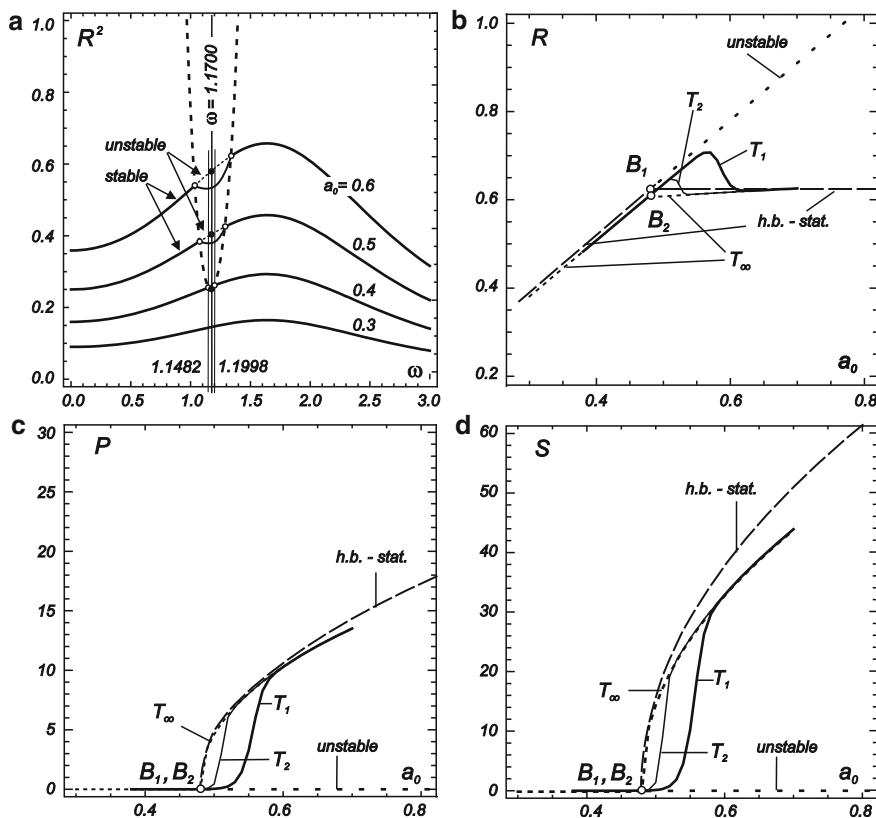


Fig. 16 Process of the stability loss and of the post-critical response; (a) instability domains for various amplitudes of excitation, (b) bifurcation diagram of the vertical response component amplitude R , (c and d) bifurcation diagrams of horizontal (rocking) components P and S

Evolution of the response amplitudes R , P , S for frequency $\omega = 1.1700$ (roughly the middle of “instability interval”) with increasing excitation amplitude a_0 is demonstrated in Figs. 16b and c. With respect to testing techniques mentioned in the third section (Lyapunov exponent, structure of the response frequency, etc.) it can be shown that Eq. 16 or 17 are suitable to evaluate response amplitudes for $\omega = 1.1700$. In Figs. 16b and c, starting from the bifurcation point B_1 , two branches evaluated by means of Eq. 17 can be found. Dotted curves represent unstable part corresponding to semi-trivial solution, while long-dashed curves (nearly horizontal for R and rising curvilinear for P , S) demonstrate stable post-critical solution.

Evaluating the system response for the same parameters and homogeneous initial conditions by means of numerical simulation, nearly the same results are provided, when a long time elapsed $T \rightarrow \infty$ (the curve T_∞) and a steady state is concerned. If the period of excitation is limited, an effect of transition from the semi-trivial until post-critical stable state proceeds, see curves T_1 and T_2 in Fig. 16b–d. For short

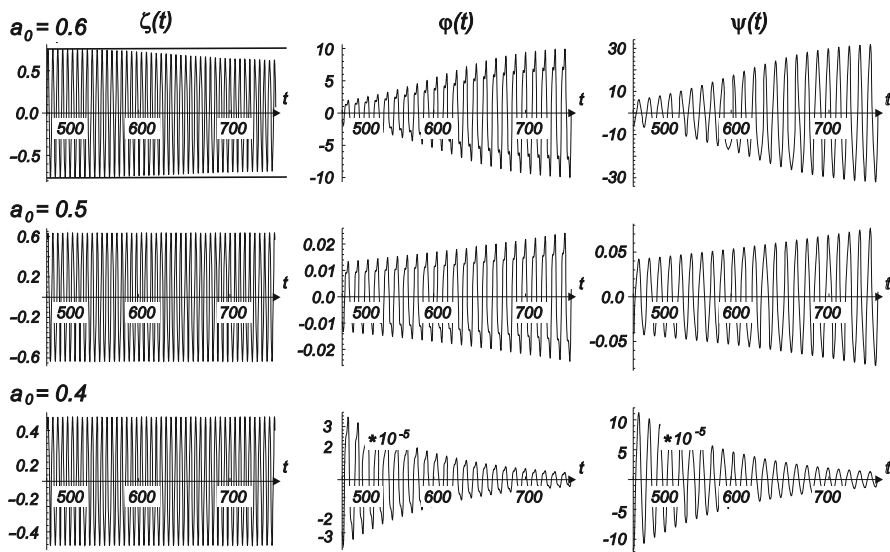


Fig. 17 Time history of the response components $\zeta(t), \varphi(t), \psi(t)$ for increasing excitation amplitude

excitation period (curve T_1) the excitation amplitude a_0 can significantly exceed the bifurcation point B_2 and the response has still the semi-trivial character, although increasing a_0 even more, the response is approaching to the steady state (long-dashed curves, T_∞). The medium time of excitation leads to some intermediate response type (T_2 curves). It can be ascertained that for time limited excitations, e.g. earthquake attack, higher excitation amplitudes can be admissible than those obtained for the steady state vibrations.

Detailed time history of all three response components for amplitudes $a_0 \in (0.4, 0.6)$ discussed above is given in Fig. 17. Asymptotical approaching of the horizontal response amplitudes $\varphi(t), \psi(t)$ to non-trivial constant are obvious. This constant in case of $a_0 = 0.4$ is rather small and the result doesn't differ from the semi-trivial solution too much.

7 Conclusions

Non-linear auto-parametric system with three degrees of freedom has been investigated. The aim of this study was to compose a simple mathematical model which enables to assess the nonlinear post-critical dynamic response of a slender vertical structure on an elastic subsoil exposed to strong vertical excitation. The existence of certain intervals of the excitation frequency has been proved where semi-trivial solution loses its dynamic stability and strong horizontal response components become decisive from the point of view of various standards. In post-critical state

mainly bending effects in the structure foundation and deformability of the vertical console are dominant. Various forms and internal structure of stability limits with respect to principal system parameters have been found. Post-critical states have been thoroughly analysed with respect to excitation frequency and excitation amplitude. Various types of post-critical regimes of the system response can be encountered beyond the stability limit. They include deterministic and chaotic types. Both of them enable steady state, quasi-periodic and irregular time history. Limit cycle portraits can have a single as well as multi-harmonic form. Post-critical response above the bifurcation point depends significantly on time when excitation is applied. The response amplitudes can remain in acceptable limits when adequate excitation (frequency and amplitude) acts throughout a short time interval. It is mostly not the case when the excitation lasts an infinite time and the response is approaching to the steady state.

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References

1. Abarbanel, H.D.I., Brown, R., Kadtke, J.B.: Prediction in chaotic nonlinear systems: Methods for time series with broadband Fourier spectra. *Phys. Rev. A* **41**, 1742 (1990)
2. Arnold, V.I.: *Mathematical Methods of Classical Mechanics*. Springer, New York (1978)
3. Bajaj, A.K., Chang, S.I., Johnson, J.M.: Amplitude modulated dynamics of a resonantly excited autoparametric two degree of freedom system. *Nonlinear Dynam.* **5**, 433–457 (1994)
4. Baker, G.L.: Control of the chaotic driven pendulum. *Am. J. Phys.* **63**, 832–838 (1995)
5. Benettin, G., Galgani, L., Giorgilli, A., Strelcyn, M.: Lyapunov characteristic exponents for smooth dynamical systems and Hamiltonian systems: A method for computing all of them. Part 2, Numerical application. *Meccanica* **15**, 21 (1980)
6. Burton, T.D.: Non-linear oscillator limit cycle analysis using a time transformation approach. *Int. J. Non-linear Mechanics* **17**(1), 7–19 (1982)
7. Cameron, T.M., Griffin, J.H.: An alternating frequency/time domain method for calculating the steady state response on nonlinear dynamic systems. *ASME J. Appl. Mech.* **56**, 149–154 (1989)
8. Chetayev, N.G.: *Stability of Motion* (in Russian). Nauka, Moscow (1962)
9. Chossat, P., Lauterbach, R.: *Methods in Equivalent Bifurcations and Dynamical Systems*. World Scientific, Singapore (2000)
10. El Rifai, K., Haller, G., Bajaj, A.K.: Global dynamics of an autoparametric spring-mass-pendulum system. *Nonlinear Dynam.* **49**, 105–116 (2007)
11. Guckenheimer, J., Holmes, P.: *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*. Springer, New York (1983)
12. Hammel, S.M.: A noise reduction method for chaotic systems. *Phys. Lett. A* **148**, 421 (1990)
13. Hatwal, H., Mallik, A.K., Ghosh, A.: Forced nonlinear oscillations of an autoparametric system. *ASME J. Appl. Mech.* **50**(3), 657–662 (1983)
14. Haxton, R.S., Barr, A.D.S.: The autoparametric vibration absorber. *ASME J. Appl. Mech.* **94**, 119–125 (1974)
15. Iooss, G., Adelmeyer, M.: *Topics in Bifurcation Theory and Applications*. World Scientific, London (1992)

16. Lee, W.K., Hsu, C.S.: A global analysis of an harmonically excited spring-pendulum system with internal resonance. *J. Sound Vib.* **171**(3), 335–359 (1994)
17. Lichtenberg, A.J., Lieberman, M.J.: *Regular and Stochastic Motion*. Springer, New York (1983)
18. Manuca, R., Savit, R.: Stationarity and nonstationarity in time series analysis. *Physica D*, **99**, 134 (1996)
19. Mazzino, A., Musacchio, S., Vulpiani, A.: Multiple-scale analysis and renormalization for preasymptotic scalar transport. *Phys. Rev. E*, **71**, 011113 (2004)
20. Moser, J.: *Stable and Random Motions in Dynamical Systems*. Princeton University Press, Princeton (1973)
21. Murphy, K.D., Bayly, P.V., Virgin, L.N., Gottwald, J.A.: Measuring the stability of periodic attractors using perturbation-induced transients: applications to two non-linear oscillators. *J. Sound Vib.* **172**(1), 85–102 (1994)
22. Nabergoj, R., Tondl, A.: A simulation of parametric ship rolling: Effects of hull bending and torsional elasticity. *Nonlinear Dynam.* **6**, 265–284 (1994)
23. Náprstek, J.: Non-linear self-excited random vibrations and stability of an SDOF system with parametric noises. *Meccanica* **33**, 267–277 (1998)
24. Náprstek, J., Fischer, C.: Auto-parametric semi-trivial and post-critical response of a spherical pendulum damper. In: *Proceedings of the 11th Conference on Civil, Structural and Environmental Engineering Computing* (B.H.V. Topping ed.). Civil-Comp Press, Malta St.Julian, CD, paper #CC2007/2006/0020, pp. 18 (2007)
25. Ott, E.: *Chaos in Dynamical Systems*, 2nd edn. Cambridge University Press, Cambridge (2002)
26. Ren, Y., Beards, C.F.: A new receptance-based perturbative multi-harmonic balance method for the calculation of the steady state response of non-linear systems. *J. Sound Vib.* **172**(5), 593–604 (1994)
27. Rosenstein, M.T., Collins, J.J., De Luca, C.J.: A practical method for calculating largest Lyapunov exponents from small data sets. *Physica D* **65**, 117 (1993)
28. Sandri, M.: Numerical calculation of Lyapunov exponents. *Mathematica J.* **6**, 78–84 (1996)
29. Schuster, H.G.: *Deterministic Chaos: An Introduction*, 4th edn. VCH, Weinheim (2005)
30. Skokos, C.: The Lyapunov characteristic exponents and their computation. Cornell University arXiv, <http://arxiv.org/abs/0811.0882v2> (2009)
31. Thompson, J.M.T., Stewart, H.B.: *Nonlinear Dynamics and Chaos*, 2nd edn. Wiley, Chichester (2002)
32. Tondl, A.: *Quenching of Self-Excited Vibrations*. Academia, Prague (1991)
33. Tondl, A.: To the analysis of autoparametric systems. *Z. Angew. Math. Mech.* **77**(6), 407–418 (1997)
34. Tondl, A., Ruijgrok, T., Verhulst, F., Nabergoj, R.: *Autoparametric Resonance in Mechanical Systems*. Cambridge University Press, Cambridge (2000)
35. Xu, Z., Cheung, Y.K.: Averaging method using generalized harmonic functions for strongly non-linear oscillators. *J. Sound Vib.* **174**(4), 563–576 (1994)

Probability Based Size Effect Representation for Failure in Civil Engineering Structures Built of Heterogeneous Materials

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Delphine Brancherie, and Sergiy Melnyk

Abstract In this work we discuss the failure analysis of civil engineering structures built of heterogeneous materials. The material heterogeneities call for a detailed representation of typical micro or meso-structure, which can be provided with the structured FE mesh representation as illustrated herein for 2D case and two-phase material. The building-block of such a representation is the constant stress triangular element that can contain two different phases and phase interface, along with all modifications needed to account for inelastic behavior in each phase and the corresponding inelastic failure modes at the phase interface. We further show by numerical simulations that the proposed structured FE mesh approach is much more efficient than the non-structured mesh representation. This feature is of special interest for probabilistic analysis, where a large amount of computation is needed in order to provide the corresponding statistics. One such case of probabilistic failure analysis is also considered in this work, where the geometry of the phase interface remains uncertain since it is obtained as the result of the Gibbs random process. This computation is further used to provide the appropriate probabilistic description of material parameters of phenomenological model of localized failure in terms of correlated random fields. Subsequent Monte Carlo computations of failure phenomena in simple tension test performed with such probabilistic phenomenological model clearly show the capability of presented approach to recover the size effects anywhere within a range between the two classical bounds which are Continuum Damage Mechanics and Linear Fracture Mechanics.

1 Introduction and Motivation

In this paper, we address the problem of modelling the size effect encountered in failure phenomena of engineering structures built of quasi-brittle materials. The approach to failure analysis we propose is placed within stochastic framework, which

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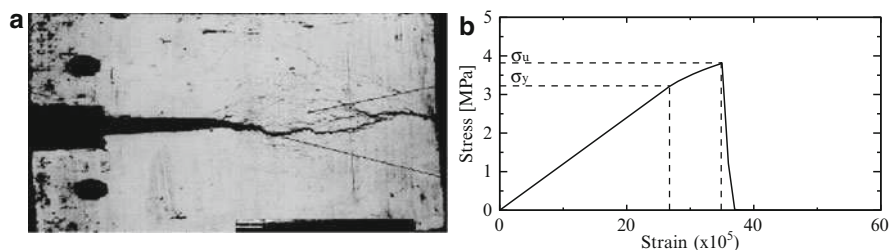


Fig. 1 Tensile test behavior: (a) Typical quasi-brittle failure pattern in engineering structure, (b) Quasi-brittle material 1D model

provides a very good basis for taking into account the intrinsic randomness of the heterogeneities of real building materials: concrete, mortar, soils or any other geomaterial. Such materials have a particular mechanical behavior, known as quasi-brittle, which can be seen as a sub-category of softening materials (see [11]). A typical failure pattern we should be able to represent contains the fracture process zone (FPZ) along with the macro-crack that is a final threat for the structural integrity (see Fig. 1a). In the context of simple 1D model interpretation, this behavior can be described with four material parameters (e.g. see [12] or [4]): Young's modulus E , the yield stress σ_y which induces micro-cracking or the FPZ creation and the failure stress σ_u which induces macro-cracking after the sudden coalescence of the micro-cracks leading to a softening behavior (see Fig. 1b). The last parameter is the fracture energy G_f [$\text{J}\cdot\text{m}^{-2}$] which represents the amount of energy necessary to create and open a macro-crack.

Several theories exist on how to model the failure in quasi-brittle materials and most of them link the micro-cracks coalescence phenomena to size effect, a dependency on the size of a structure to its failure load. The aim of all those theories is to hook up together the continuum damage mechanic (CDM) where the failure stress does not depend on the size of the structure to the linear fracture mechanic (LFM) where size effect appears naturally as the logarithm of the failure stress depends linearly on the logarithm of the size of the structure (see Fig. 13). It can be experimentally demonstrated that even if purely brittle materials follow LFM, quasi-brittle materials do not and follow a non linear relationship between the two previous logarithms of the failure stress and the size of the structure. These materials exhibit a different size effect than the one encountered for purely brittle materials.

Numerical analysis of failure processes of quasi-brittle structures raises many important modelling issues, chief among them how to account for heterogeneities of real materials and how fine should be the chosen scales. Cement-based materials, such as concrete or mortar, can be considered at different scales, depending on the objectives and the physical mechanisms that are important to account for. Considering nowadays computational resources for typical engineering applications, most frequently we ought to perform an analysis at the structure scale or macro-scale. At this scale cement-based materials might be considered as homogeneous, and their properties obtained by using the key concept of Representative Volume Element

(RVE, see [3, 17]) to retrieve phenomenological models of inelastic behavior (e.g. see [1, 11, 39]). Those models are well known for their robustness and lead to relatively small computational cost. Due to two these main points, phenomenological approaches are widely spread. On the other hand, such models are based on a set of “material” parameters which need to be identified (e.g. see [20]), mainly from experiments and providing unique load paths and boundary conditions. Hence this “natural” methodology leads to a set of parameters which is linked to the chosen load-path. Being not adapted to another path, it’s hardly possible to get any predictive features from those phenomenological macro-model. The main reasons for this is that the macro-scale is not the right scale to consider with the aim to model failure of heterogeneous materials. Many authors tried to overcome this major drawback by furnishing micro-mechanical bases to the macroscopic model set of parameters (see [21, 23]) and provide more predictive macro-scale models. One possible way to achieve such goal is to adopt homogenization methods which are leading to accurate results for linear problems. In presence of non-linearities such methods are not capable to provide good estimates for the effective (macroscopic) properties (see [9]). Moreover, such approach does not take into account the inherent uncertainties attached to heterogeneous materials and structures.

At finer scale than the macroscopic one, cement based materials appear to be heterogeneous exhibiting an important variability. This variability might be viewed from the geometrical point of view, considering the arrangement (positions, shapes) of the different phases. In this work we propose to take account for the meso-scale variability in order to: First compute the macroscopic (effective) properties statistics for a porous media made of a non-linear matrix. Second show how these statistics (mainly the correlation length through the covariance function) might be used at the macroscopic level to model particular features of cement-based materials such as size effects. The key point is that the material parameters at meso-scale are assumed to be deterministic, so that the variability is only related to the size and the positions of the voids in the porous media. In order to solve this stochastic problem and compute the statistical moments for the response quantities, we employ the Monte-Carlo method within a distributed software environment. This stochastic integration method is based on many evaluations of the meso-structures responses thus leading to a time-consuming process. Moreover, as the error can directly be evaluated in terms of the number of realizations, it is necessary to choose a relatively small discrete problem, even in the case of complex meso-structures. To achieve this we propose a model based on a regular mesh which is not constrained by the physical interfaces. This model relies on classical CST elements, whose kinematics description is enriched by the use of strain and displacements discontinuities in order to represent two phases.

The outline of this chapter is as follows: in Sect. 2, we present the meso-scale level description, with the plasticity model employed and the structured mesh representation, leading to fast computing of non-linear response without any remeshing. In Sect. 3 we describe the stochastic problem, the geometrical description process for defining the meso-structure and the stochastic integration method. Section 4 presents the results obtained for the SRVE size, as well as the corresponding

macroscopic properties statistics. Finally Sect. 5 deals with size effect modelling based on second order correlated macroscopic properties fields. Concluding remarks are given in Sect. 6.

2 Meso-Scale Model of Material Heterogeneities with Deterministic Material Parameters

Meshing is one of the major issue in modelling heterogeneous materials. The possibly high number of phases and their complex shapes frequently might lead to a quite high number of degrees-of-freedom and also quite distorted meshes. Moreover, the meshing process itself might consist in a complex and time-consuming algorithm. The objective of this first part is to show how to employ structured meshes in order to simplify the meshing process of heterogeneous materials. Hence this section presents the main ideas leading to regular meshes which are not constraint by the physical interfaces between the different phases. The key ingredients to provide such models are field discontinuities introduced inside the elements in which the physical interfaces are present. These kinematics enhancements might be developed within the framework of the Incompatible Modes Method (see [15,37]), and require a dedicated solution algorithm which is illustrated next.

2.1 Structured Mesh and Element Kinematics Enhancements

At meso-scale, we consider a heterogeneous material in 2D built of different phases and we assume that each of these phases is described by the inclusions positions and shapes. In order to model such material with a structured mesh, Fig. 2 shows a typical 3-nodes triangular finite element representing two phases. Those two phases are introduced through two types of discontinuities (see [14]), namely a discontinuity of the strain field and a discontinuity of the displacement field, both of them lying at the same position (prescribed by the known physical interface between the

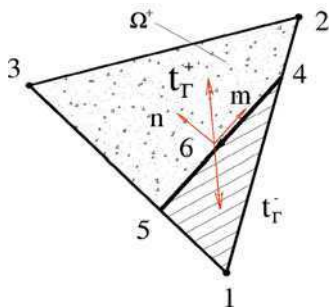


Fig. 2 Two phase 3 node triangular element, interface position and the corresponding sub-domains

two phases). The strain discontinuity permits the proper strain representation of two different sets of elastic properties corresponding to each phase. The displacement discontinuity leads to the possibility to model debonding or any failure mechanism at the interface. For the latter, two failure mechanisms are considered: one corresponding to the opening of the crack in the normal direction and the second one to the sliding in the tangent direction (see [32]). Both of these discontinuities are introduced by using the Incompatible Modes Method (see [15, 37]). The key advantage of this method is to lead to a constant number of global degrees-of-freedom.

Both of those kinematics enhancements are added on top of the standard CST element (Fig. 2). Hence this element is divided into two parts by introducing an interface whose position is obtained by the intersection of the chosen structured mesh with the inclusions placed within the structure. The domain Ω^e of the standard 3-node constant stress triangle (CST) element is thus divided into two sub-domains Ω^{e-} and Ω^{e+} . One of the most important and well-known features of strong (displacement field) discontinuities models is their capability to be independent from the mesh, even for softening laws. This ability lies in the fact that the dissipation process occurs on a line (i.e. the interface) and not in the whole volume. However, different elastic-plastic or elastic-damage behaviors laws, with positive hardening, might be chosen for each of the two sub-domains split by the interface, with different elastic properties (see [13]).

It is worth to note that the strain field discontinuity is always present, due to the different elastic constants between the two phases. In contrast and because of representing a failure mechanism between the two phases, the displacement field discontinuity needs to be activated according to some chosen failure criterion.

Introducing those discontinuities requires to enhance the kinematics of the element by using two incompatible modes. Thus, the displacements field might be written as follows:

$$\mathbf{u}^h(\underline{x}, t) = \sum_{a=1}^3 N_a(\underline{x}) \mathbf{d}_a(t) + \mathbf{M}_I^\alpha(\underline{x}) \alpha_I(t) + \mathbf{M}_I^\beta(\underline{x}) \beta_I(t) + \mathbf{M}_{II}(\underline{x}) \alpha_{II}(t) \quad (1)$$

This expression contains four terms: the first one provides a constant strain field inside the element (as the classical CST element does). The second and third terms both represent jumps in the displacements field, in the normal and the tangential directions. Finally, the last part provides the strain field discontinuity. All the strain and displacement enhancements are limited to a single element only; the latter provides much better basis for constructing robust operator split analysis from the X-FEM method. The shape functions $\mathbf{M}_I(\underline{x})$ for the first incompatible mode (see Fig. 3a) corresponding to the displacements field discontinuity for both normal and tangent directions (see [12]) might be written as:

$$\mathbf{M}_I(\underline{x}) = H_{\Gamma_S}(\underline{x}) - \sum_{a \in \Omega^+} N_a(\underline{x}) \quad (2)$$

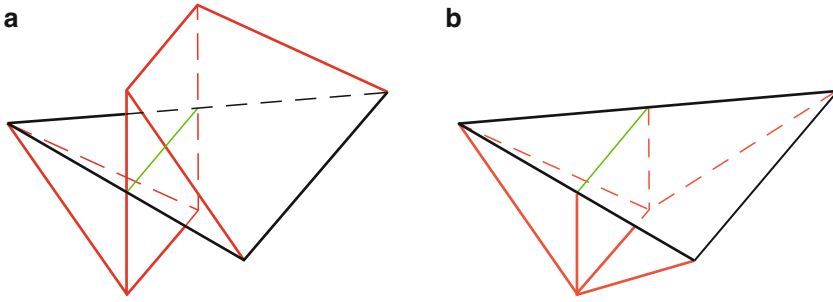


Fig. 3 Incompatible modes corresponding to displacements (a) and strain (b) discontinuities for CST element

where N_a represents the normal CST shape functions element and H_{Γ_S} the Heaviside function placed at the interface position. The shape function $\mathbf{M}_{\Pi}(\underline{x})$ which provides the jump in the strain field is shown in Fig. 3b.

Considering the displacement interpolation (1), the strain field might be written as:

$$\varepsilon^h(\underline{x}, t) = \mathbf{B}\mathbf{d} + \mathbf{G}_{\Pi}\alpha_{\Pi} + (\mathbf{n}^T \otimes \mathbf{n})\mathbf{G}_{\mathbf{I}_r}^{\alpha}\alpha_{\mathbf{I}} + \frac{1}{2}[\mathbf{n}^T \otimes \mathbf{m} + \mathbf{m}^T \otimes \mathbf{n}]\mathbf{G}_{\mathbf{I}_r}^{\beta}\beta_{\mathbf{I}} \quad (3)$$

where $\mathbf{B}(\underline{x})$ are the well known CST element strain-displacement matrix (e.g. see [39]) and $\mathbf{G}_{\mathbf{I}_r}(\underline{x})$ contains the derivatives of the first incompatible mode. Finally, in (4), $\mathbf{G}_{\Pi}$ is the matrix containing the derivatives of the second shape function $\mathbf{M}_{\Pi}(\underline{x})$.

2.2 Operator Split Solution Procedure for Computing Interface Failure Modes

Deriving from the Incompatible Modes Method for the two kind of discontinuities added on the top of the classical CST element (strain field and displacement field), the total system to be solve consists of four equilibrium equations, with (4a) as the global equilibrium equation and (4b) to (4d) are corresponding to the local ones. It is worth to remind that Eqs. 4b and 4c have to be solved only in case of activation of the displacement discontinuity in the normal or the tangent direction.

$$\begin{cases} A_{e=1}^{nel}[f^{int} - f^{ext} = 0] \\ \mathbf{h}_{\mathbf{I}}^{\alpha,e} = 0 \\ \mathbf{h}_{\mathbf{I}}^{\beta,e} = 0 \\ \mathbf{h}_{\Pi}^e = 0 \end{cases} \implies \begin{cases} \int_{\Omega^e} \mathbf{B}^T \sigma d\Omega - \int_{\Omega^e} \mathbf{N}^T b d\Omega = 0 \\ \int_{\Omega^e} \mathbf{G}_{\mathbf{I}_r}^{\alpha,T} \sigma d\Omega = 0 \\ \int_{\Omega^e} \mathbf{G}_{\mathbf{I}_r}^{\beta,T} \sigma d\Omega = 0 \\ \int_{\Omega^e} \mathbf{G}_{\Pi}^T \sigma d\Omega = 0 \end{cases} \quad (4)$$

The consistent linearization (e.g. see [11]) of this set of equations leads to the linear system, in the matrix form:

$$\begin{bmatrix} \mathbf{K}^e & \mathbf{F}_{I_r}^{\alpha,e} & \mathbf{F}_{I_r}^{\beta,e} & \mathbf{F}_{II}^e \\ \mathbf{F}_{I_r}^{\alpha,e^T} & \mathbf{H}_I^{\alpha,e} & \mathbf{F}_H^e & \mathbf{F}_S^{\alpha,e} \\ \mathbf{F}_{I_r}^{\beta,e^T} & \mathbf{F}_H^{e^T} & \mathbf{H}_I^{\beta,e} & \mathbf{F}_S^{\beta,e} \\ \mathbf{F}_{II}^{e,T} & \mathbf{F}_S^{\alpha,e^T} & \mathbf{F}_S^{\beta,e^T} & \mathbf{H}_{II}^e \end{bmatrix}_{n+1}^{(k)} \begin{pmatrix} \Delta d \\ \Delta \alpha_I \\ \Delta \beta_I \\ \Delta \alpha_{II} \end{pmatrix}_{n+1}^{(k+1)} = \begin{pmatrix} -r \\ -\mathbf{h}_I^{\alpha,e} \\ -\mathbf{h}_I^{\beta,e} \\ -\mathbf{h}_{II}^e \end{pmatrix}_{n+1}^{(k)} \quad (5)$$

The expanded form for each block can be found in [10].

The operator split strategy consists in first solving the local equations of system (4) (namely Eqs. 4b–4d) at each numerical integration point and for fixed global degrees-of-freedom values. The second step is then to carry out static condensations (e.g see [36]). These static condensations leads to the effective stiffness matrix (see [10, 31] and thus the last step is to solve the global system of equations (4) to obtain the updated value of the displacement field $d_{n+1}^{(k+1)} = d_{n+1}^{(k)} + \Delta d_{n+1}^{(k+1)}$

$$\widehat{\mathbf{K}}_{n+1}^{(k)} \cdot \Delta d_{n+1}^{(k+1)} = -r_{n+1}^{(k)} \quad (6)$$

One of the key point to note is that the total number of global unknowns remains the same as with the standard CST element which is the major advantage of Incompatible Modes Method. Simple illustrative examples dealing with the use of structured meshes might be found in [10].

2.3 Comparison Between Structured and Unstructured Mesh Computations

Here we aim to make a comparison between structured and unstructured meshes in order to assess the capability for both cases to get very close results. For this we consider a porous material made of a perfectly plastic matrix with circular voids of different sizes. The first case (Fig. 4a) presents an exact mesh obtained by using the software GMSH. Obviously in this case each element contains only one phase (namely the matrix or the “voids”). Moreover several elements are strongly distorted and they exhibit quite different sizes. For these two reasons the stiffness matrix is poorly conditioned. The second case (Fig. 4b) relies on a structured mesh which is based on a regular grid. In this case, the elements needs to represent two phases to model the inclusions and we adopt the strategy presented at the beginning of this section. Figure 4 shows the axial displacement contour plot (with an amplification factor of 100) for both unstructured and structured meshes. Figure 5 plots the corresponding macroscopic axial reactions displacement curve.

We show that both cases are providing very close results, but with a gain of computing time in favor of the structured mesh strategy (this point is mainly due to the tangent matrix optimal conditioning). Combined to a meshing process which

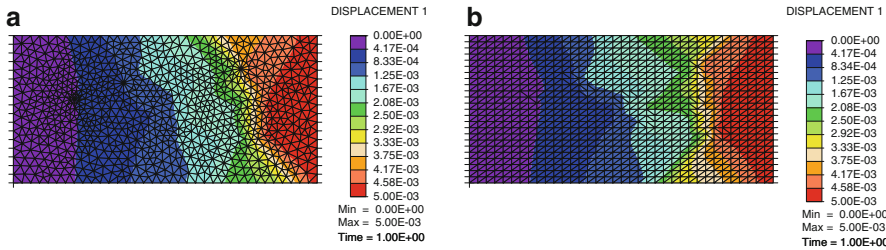


Fig. 4 Longitudinal displacement contour plot corresponding to max.load for adaptive mesh (a) and regular mesh (b)

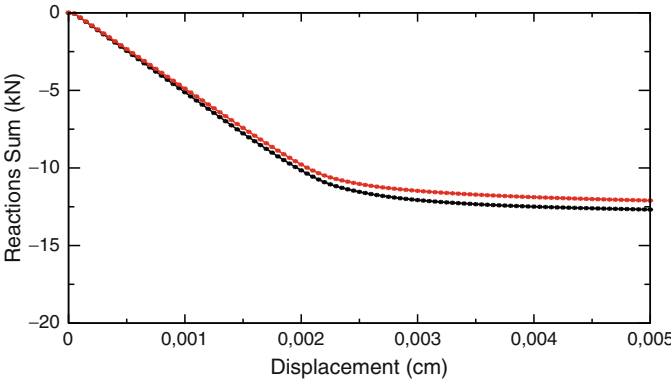


Fig. 5 Reactions sum versus displacement curve (black unstructured mesh, red structured mesh)

is much easier, the structured mesh way appears to be a good and accurate method to model heterogeneous material, especially in the context of many realizations that have to be analyzed. This last point is one of the key issues considering probabilistic aspects for heterogeneous materials.

3 Probability Aspects of Inelastic Localized Failure for Heterogeneous Materials

At finer scale than the macroscopic one, cement-based materials obviously appear to be heterogeneous. As an example, at this meso-scale mortars are made of three phases: two solid ones (the grains and the cement paste) and voids. It is well-known from experimental data that macroscopic properties of such materials are strongly linked to the (at least) meso-scale constituents. In ([38]) the authors gathered some experimental results showing the very important decreasing of macroscopic mechanical strength (in tension or compression) along the increasing voids volume

fraction. Moreover, considering a constant porosity, the voids shapes and positions also have a major influence on the macroscopic properties, especially for small specimens. This key point is linked to the statistical RVE size (see e.g. [17]) which size has to be determined along a prescribed macroscopic error tolerance. The main objective of this section is then to illustrate the possibilities provided by the use of structured mesh representation and the efficient computation capabilities of the proposed model for dealing with random heterogeneities. To that end, we consider herein a porous material, typical of mortars at a meso-scale level. At this scale we assume that such material is characterized by a two-phase micro-structure with a solid phase and a fluid phase. The former will be referred as the “matrix” and the latter is supposed to represent the voids or inclusions. Depending on the number of inclusions, their sizes and positions, the non-linear macroscopic response of such a material will vary. In other words, the macroscopic properties, such as Young’s modulus or the yield stress, will be influenced by the meso-scale geometry. Our goals here are: first to determine the statistical RVE size corresponding to such a geometry (morphological RVE); second to carry out numerically the variations of the macroscopic characteristics upon the inclusion sizes and positions. The key point for this study is that the variability introduced into the model is restricted to the specimen geometry only, whereas the mechanical characteristics of the two phases are assumed to be deterministic. To be more precise, the matrix phase is supposed to be accurately modelled by an elastic-perfectly plastic model based upon the Drucker-Prager criterion (see [8]). The voids are represented by a simple linear isotropic elasticity model with very small Young’s modulus value. In the following sections we first begin to describe the Gibbs point process, leading to the realizations of the meso-structures. We also show an example of one typical mesh obtained and the corresponding macroscopic response to a tension test. Then we turn to the stochastic integration method which has been chosen to numerically solve this problem and the corresponding Software Engineering aspects. Finally we present the methodology leading to the RVE definition and we discuss the results obtained for this stochastic problem.

3.1 Geometry Description of Material Meso-Structure

Here we describe both the process and the hypothesis leading to the meshing procedure within a rectangular domain ($3.6 \times 1.8 \text{ cm}^2$). The meso-structure geometry of such domain is here supposed to be accurately modelled by a Gibbs point process. Such point process is built on a two steps scheme. The first one is the determination of the inclusions number according to a Poisson law. The second step consists in the determination of the inclusion centers coordinates as well as the radius for each inclusion. While such a Gibbs process already naturally leads to a set of non-intersecting inclusions, we applied an even more restrictive criterion, by choosing the minimal distance between the inclusions (here equal to 2 mm). Moreover, in order to be consistent with the mesh size and the model features, the inclusions radius

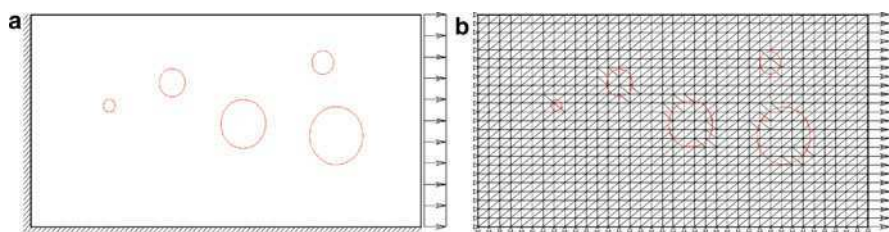


Fig. 6 Meso-structure geometry (a) and corresponding structured mesh (b)

are bounded between 1 mm and 3 mm. Figure 6 shows a particular realization of the meso-structure and the corresponding structured mesh. We can notice that each inclusion is correctly modelled by a set of discontinuities without any major distortion.

Since the material parameters are chosen to be deterministic, the statistics of the macroscopic response depends on the meso-structure geometry only, defined by the voids volume fraction and consequently the voids radius and centers positions. Thus the macroscopic problem is stochastic and requires stochastic integration method which is presented in the next section.

3.1.1 Stochastic Integrals Computations

As we just mentioned, the positions and sizes of the voids in the matrix are described by a discrete random field, based on a Gibbs point process. In a general sense, a 2D random point process might be viewed as a finite set of random variables, which are indexed by the spatial coordinates vectors in R^2 . Thus the meso-structure geometry is defined as a random field, which implies that every solution computed by the mechanical model is also a random field (e.g., the structure displacement at a fixed point is a random variable). In this study, we are interested in characterizing the macroscopic mechanical properties of our structure. To achieve this goal, we use a global approach which consists in identifying the material properties governing the global behavior of the structure. More precisely, we aim to determine the effective global material properties by the corresponding identification of the global response computed by the Finite Element model. Therefore, since the global responses (displacement and reactions) are random variables, the global material properties we aim to identify, such as the Young modulus or the yield stress, are also random variables.

Probabilistic characterization of the macroscopic mechanical properties can be viewed as describing the probabilistic law followed by each of these properties. Two approaches can be drawn to find a probabilistic law describing a random phenomena. The first one, so-called frequentist approach [19], is based on statistical tests, like the χ^2 test for the Gaussian probability law. Results of these tests are error margins that evaluate how the outcomes of the given random phenomena fit with respect to a given probability law. The second, so-called Bayesian approach

[16], is trying to use all the available information along with the maximum entropy theory (see [30, 34]) in order to provide the most general probability law for a given state of information; thus, to fully describe this probability law, the statistical moments of different orders have to be computed. In this work, the second approach is chosen. The macroscopic material properties we tend to characterize are all defined on the positive real line. Moreover we assume that they can be given a mean value and a finite standard deviation. On the basis of such information, the maximum entropy theory leads to the most general probability law for this case in terms of the log-normal distribution, which is fully described by its computed mean value and standard deviation.

Consequently, in order to characterize the macroscopic mechanical properties using the Bayesian approach, the first two statistical moments of each of these properties have to be computed. The statistical moment of any random variable is an integral of a functional of this random variable over a probability space. Hence, an efficient numerical tool to compute such integral in multi-dimensional space is required. Rather than high order quadrature rules like Smolyak algorithm [33], we use here a simple direct integration algorithm which is Monte Carlo simulation [5]. The basic idea of Monte Carlo simulation is to approximate the integrals of a functional of a random variable by a weighted sum of realizations of this random functional. Let ξ be a random variable defined on some probability space (Ω, B, P) , where Ω is the space of event, B is a σ -algebra built on Ω and P a probability measure. Any defined moment of ξ can be written as $\int_{\Omega} f(\xi(\omega))dP(\omega)$. The simple Monte Carlo algorithm consists in approximating this integral as a finite weighted sum of realizations $f(\xi(\omega_i))$, each computed at a randomly independent chosen point ω_i in Ω , multiplied by the corresponding weights $\frac{1}{N}$ (with N the given number of realizations)

$$\int_{\Omega} f(\xi(\omega))dP(\omega) \approx \frac{1}{N} \sum_{i=1}^N f(\xi(\omega_i)) \quad (7)$$

For this kind of numerical integration, the convergence rate can be *a priori* computed thanks to the central limit theorem [22]. We can find the error estimate which is proportional to the standard deviation of $f(\xi)$ over $\sqrt{N}$, N being the number of evaluations of $f(\xi)$. As each realization of the Gibbs process is stochastically independent from the others, this method can be directly applied here and further more parallelized using an appropriate software environment to eliminate the main drawback of Monte Carlo algorithm, the slow-rate convergence. In this case, where no correlation exists in the geometrical space, no other tools such as Karhunen-Loève expansion is required (see [7, 22]).

The software architecture used here is based on the software component technology and the middleware CTL [24], which provides the adequate network environment to enable code communication under a prescribed protocol and more generally code coupling. The basic idea of software component technology is to divide a software framework into several tasks and then to implement software components, each of them being able to carry out this particular task. Existing software can be

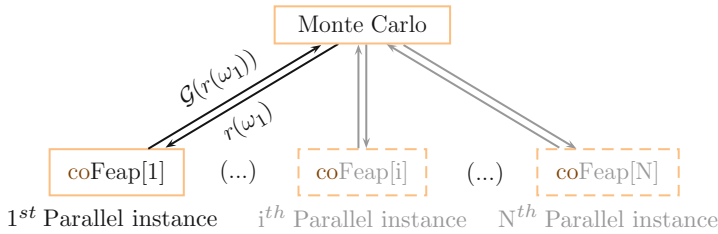


Fig. 7 Parallel software architecture for Monte Carlo simulations using coFeap component

turned into a component by defining an interface through which the communication will be channelled. Implementing a component from for a pre-existing program consists in coding a set of methods that other software can call through this interface. In the case of Monte Carlo simulations, two different tasks can be drawn. One is to generate a Gibbs process and to transfer this result defining the inclusions geometry in a structured mesh. The other is to run a computation with this given geometry within the mechanical model defined in the first section. Based on the FEM code Feap [39], a CTL software (e.g. see [25, 26]) component named coFeap has been derived [18]. The second component in charge of the geometry generation (the so-called client in Fig. 7) will ask for several runs of the coFeap component at the same time, each of them using a different meso-structure realization.

Further details on the use of this parallel framework and results are presented in the following section.

4 Probabilistic Characterization of Two-Phases Material at Macro-Scale

4.1 Determination of Statistical RVE Size

As we already mentioned in the introduction part, macroscopic models are usually based on the concept of RVE. Here we focus on the notion of Statistical RVE (see [27]) leading to a volume element large enough to assure that its macroscopic properties are assumed to be deterministic, up to a certain tolerance. Obviously such size strongly depends on the properties to be considered and we restrain our analyze to geometrical properties, namely the voids volume fraction. Following [17], the methodology adopted consists in estimate the voids volume fraction mean $\bar{\rho}$ and variance σ_ρ^2 along different domain sizes. In order to get those estimates we used the Monte Carlo framework presented before considering a set of 10,000 realizations for each volume fraction and domain size. As a result, Fig. 8 shows the 0.95 confidence interval along the domain size. This confidence interval is defined for one realization as $[\bar{\rho} - 1.96\sigma_\rho; \bar{\rho} + 1.96\sigma_\rho]$.

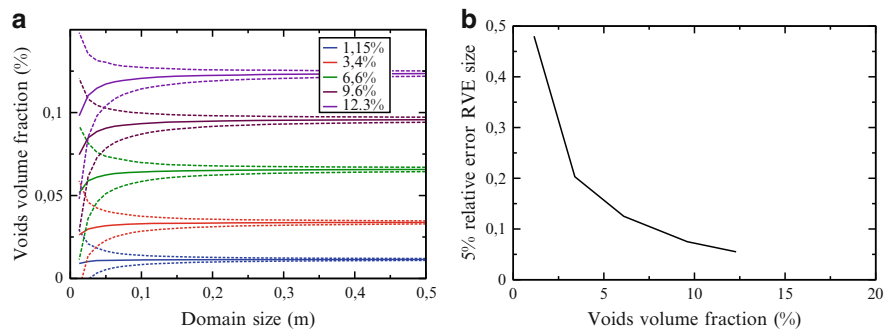


Fig. 8 (a) Voids volume fraction confidence interval (0.95) along domain size and (b) 5% relative error morphological RVE size

Table 1 SRVE size for different mean voids volume fractions and 5% relative error

ρ (%)	1.15	3.4	6.1	9.6	12.3
SRVE size (m)	0.48	0.203	0.125	0.075	0.055

Table 1 as well as Fig. 8b show the SRVE size corresponding to 5% relative error for different voids volume fraction. Clearly the SRVE size is decreasing with an increasing porosity. Typical porosity values for mortars are in the 5–10% range and so leading to a 0.07–0.15 m morphological SRVE estimate range. As we mentioned before, although the same methodology might be followed in different cases, this estimate does not provide any information about SRVE size linked to any non-linear mechanical properties (e.g. macroscopic yield stress).

4.2 Simple Tension Test: Numerical Results and Discussion

By using the stochastic numerical integration method detailed in the previous section, we performed $Z = 10,000$ integration points, each of them corresponding to an independent meso-structure realization. These Monte-Carlo integration points have been distributed on 9 processors and we shall present here the different results.

The first point to be mentioned deals with the voids volume fraction for each meso-structure geometry. To some extent those data might be viewed as the “input” parameters according to the stochastic integration method. We recall here that each meso-structure realization is built by using a modified Gibbs point process with inclusions radius bounded between 1 and 3 mm. Figure 9 shows the voids volume fraction (ratio of the voids volume versus the total volume) histogram corresponding to the Z realizations. The associated mean value is 6.26% and the standard deviation 3.59%.

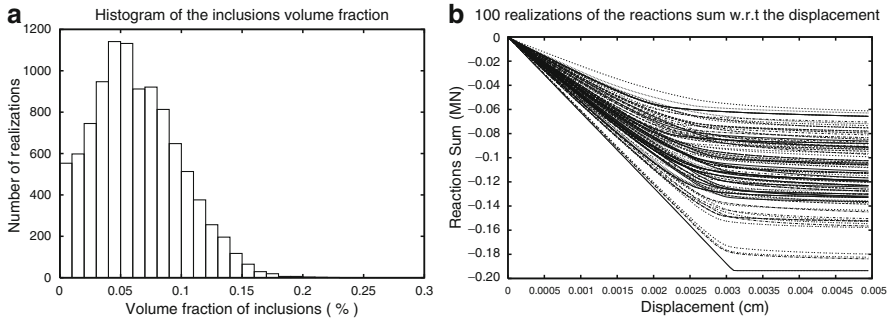


Fig. 9 (a) Voids volume fraction histogram (b) 100 realizations sample results

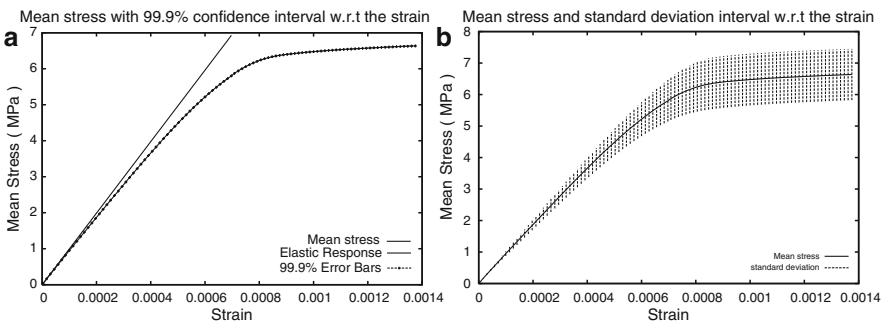


Fig. 10 Macroscopic mean stress w.r.t the macroscopic strain (a) 0.999 confidence interval (b) standard deviation

The stochastic integration process is leading to a set of Z axial reaction force-displacement diagrams. Figure 9 shows a subset of 100 realizations. It is worth to recall again that the variability shown by this sample is due only to the meso-structure geometry variability (the material parameters are assumed to be deterministic and so constant along the realizations). Moreover we shall note that some meso-structures included in this sample have obviously no voids. This point is directly linked to the Gibbs point process features, in particular to the discrete Poisson law leading to the inclusions number which is possibly zero.

Figure 10 shows the estimated mean macroscopic stress-strain curve as well as the 99.9% confidence interval. Since this confidence interval is quite narrow the number of integration points for the stochastic integration method is sufficient to make accurate conclusions and to provide good estimates of statistical moments. Moreover, the macroscopic stress σ and strain ε are defined as equivalent homogeneous quantities,

$$\varepsilon = \frac{\bar{u}}{L_x} \quad \sigma = \frac{\sum_i R_i}{L_y} \quad (8)$$

Table 2 Macroscopic quantities statistics

	Mean estimator	Std-dev interval
σ_u	6.63 MPa	[5.82 MPa, 7.45 MPa]
σ_y	2.49 MPa	[2.15 MPa, 2.54 MPa]
E	9.93 GPa	[9.27 GPa, 10.60 GPa]

where L_x and L_y are the size of the domain and R_i the axial reactions. This macroscopic mean curve leads to the determination of an estimate for the macroscopic mean Young’s modulus as well as to an estimate of the maximum stress mean σ_f . Table 2 summaries the statistical macroscopic estimates obtained from this numerical example. Those estimates are quite good candidates to be employ in the context of a macroscopic phenomenological model, here for 1D case. Actually these macro model parameters might be used in order to define macroscopic random fields and, therefore, a stochastic macro-model (that we would have to be solved using any stochastic integration method). It is worth to note that the only missing data in order to fully define weakly homogeneous second order random fields is the covariance function. In the most simple case, such second-order information might be given according to one scalar parameter only, the macroscopic correlation length.

5 Size Effect Representation

Size effects for quasi-brittle materials can be experimentally demonstrated at macro-scale and several ways exist dealing with its modelling. Most of them are linking the micro-cracks coalescence phenomena which consists in the failure process first step to such size effect. An extensive literature exists on that topic, from the early studies of Weibull (see [35]) dealing with infinite chains built from brittle links (theory of the weakest link), to the current two concurrent theories of Bazant on the one side and of Carpinteri on another. The first one tends to describe size effect as a deterministic theory of strength redistribution in a Fracture Process Zone (FPZ) which size is proportional to a characteristic length, that leads to energetic dissipation. At some level, the micro-cracks coalescence that induces both heterogeneous behavior and some kind of localization, is strongly intricate to size effect. Hence, a way to study the fracture of quasi-brittle material is to study size effect. Recently, Bazant has developed a new theory as a combination of this previous theory with the Weibull’s one leading to the so called energetic-statistical size effect (see [2]). An other theory melting a non-local model and a stochastic approach has been developed in [29]. On the other hand, Carpinteri’s theory is based on the study of quasi-brittle materials seen as materials with a fractal micro-structure (see [6]).

Our goal is to stress on the possibility to model size effect, taking place at macro-scale, with the use of correlated random fields for macroscopic properties. With some basic assumptions, such macroscopic random fields are defined by their marginal (point-wise) first and second moments as well as their covariance function. Adding isotropy condition, this covariance function might be parameterized using,

for example, a unique scalar value: the correlation length L_c . This length plays a key role in this context of size effects. Contrary to classical macroscopic model which are based on the RVE concept only, L_c actually defines a scale to which the whole structure size is compared. In that sense such correlated fields naturally incorporates size effects. Moreover and to some extent, such correlation length L_c might be considered as the “characteristic length” which needs to be defined using the well-known macroscopic non-local models [28]. Contrary to this characteristic length for which there is a lack of physical interpretation, the correlation length L_c as well as the marginal first and second orders moments necessary to fully defined macroscopic properties random fields, might be retrieved from a two-scales analysis like the one presented in the previous section. Thus we draw here a complete macroscopic modelling methodology, starting from properties and uncertainties at meso-scale (which appears to be the most pertinent one considering mechanical failure of cement based materials) to macroscopic behavior.

As an example of this methodology in a 1D context, we first begin to recall some of the key points dealing with random fields and the Karhunen-Loève expansion, which is one of the most efficient way of representation in a computational context. Then we turn to the macroscopic description of the model and its integration using once again the direct Monte Carlo simulations way.

5.1 *Random Fields for Material Properties and Their Karhunen-Loève Expansion*

The macroscopic 1D model we consider here is representing a three-stage failure process which is typical of cement based materials. After the elastic regime, the inelastic behavior starts considering a homogeneous micro-cracking in the so-called Fracture Process Zone (FPZ). Such phenomenon might be modelled by a volume dissipation process and phenomenological macro-models are obviously good candidates to this. Once having reached a certain loading limit, the coalescence of the micro-cracks turns to some kind of localization and macro-cracking. For this last stage (and so just before structure’s failure), phenomenological models are no more valid, mainly according two points: first they are leading to some mesh dependance which are typical of softening laws; second because of localization the RVE concept is not applicable. To overcome this drawback, many authors turned to the well-known non-local theory which consists in considering a characteristic length. Another way is to consider strong discontinuities models like the one presented in Sect. 2 for the meso-scale. For 1D context this model requires four parameters to fully describe all the failure process (see Fig. 1), namely the elastic modulus E , the yield stress σ_y , the failure stress σ_u and the fracture energy G_f .

In order to model size effect with such macro-model, the key idea is to consider its macroscopic parameters (namely σ_y and the gap $e_f = \sigma_u - \sigma_y$) as correlated random fields over the geometric space $\mathcal{D}$ and a probability space Ω . Mathematically speaking, Ω is a space of random elementary events, together with a class of subsets

$\mathfrak{S}$ of Ω (*i.e.* a σ -algebra, see [22]) to which a real number in the interval $[0, 1]$ may be assigned, the probability of occurrence, mathematically a measure P . A $\mathcal{V}$ -valued random variable r is then a function relating to each $\omega \in \Omega$ an element $r(\omega) \in \mathcal{V}$. In case the space $\mathcal{V}$ is a space of functions $\mathcal{F}(\mathcal{D})$ on a spatial domain $\mathcal{D}$, then r defines a random field. In some sense, this random field r can be viewed as an infinite family of random variables $r(x, \omega)$, assigned at each point $x \in \mathcal{D}$. Both σ_y and e_f are positive and supposed to have a finite known variance. We also assume that our solution, here the displacement along the bar is of the second order. Thus, thanks to the maximum entropy theory (see [30, 34]), these two random variables σ_y and e_f can be taken with lognormal distribution. Without loss of generality, it is convenient to consider that these two random fields are defined as non-linear transformations of Gaussian random fields γ_1 and γ_2 :

$$\sigma_y = \exp(\gamma_1) \quad \text{and} \quad e_f = \exp(\gamma_2) \quad (9)$$

γ_1 and γ_2 are fully described by their expected values and their covariances:

$$E_{\gamma_i}(x) \quad \text{and} \quad \text{Cov}_{\gamma_i}(x, y) = V_i \exp\left(-\frac{\|x - y\|_1}{L_c}\right) \quad (10)$$

In (10) above we indicated that the Gaussian random fields γ_1 and γ_2 are supposed to be weakly homogeneous with an exponential form of the covariance and the correlation length L_c . As the geometric space is of dimension one, the covariance can be drawn as a surface in a three dimension space over the two dimension space $\mathcal{D} \times \mathcal{D}$ (see Fig. 11).

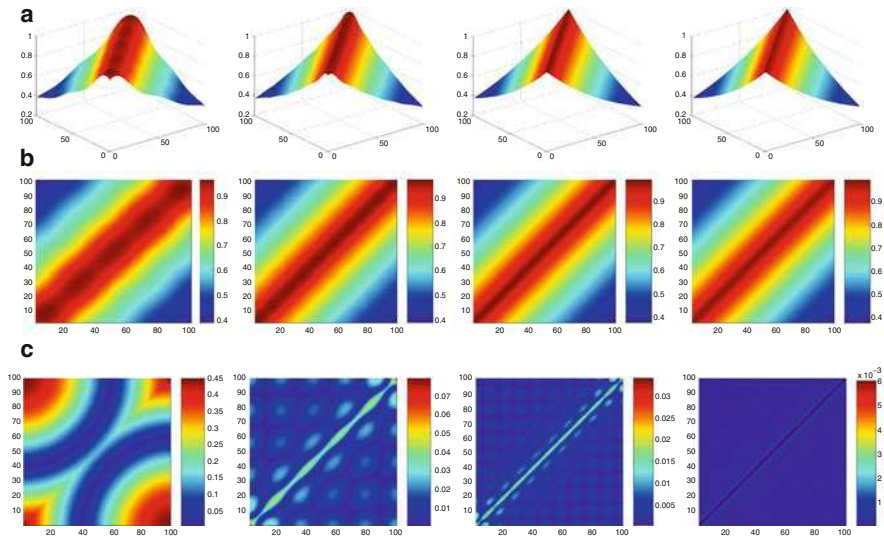


Fig. 11 Random field covariance obtained from 5, 10, 20 and 50 modes in truncated KLE expansion: (a) 3D representation, (b) 2D representation, (c) representation error

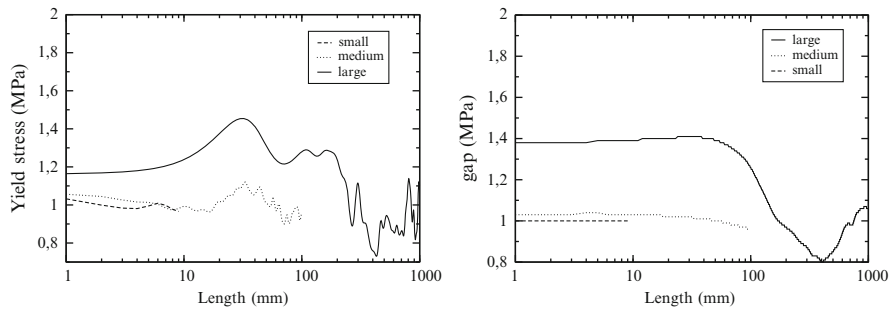


Fig. 12 Normalized realizations of correlated macro random fields σ_y and e_f

Solving for the response of such a stochastic system essentially consists in computing some response statistics (e.g. its expected value). To that end, we once again employ here the so-called Monte-Carlo method which requires to solve many deterministic systems, each of them being built with realizations $\sigma_y(\cdot, \omega_i)$ and $e_f(\cdot, \omega_i)$ (see Fig. 12) of the random fields σ_y and e_f . In order to provide these realizations an effective computational representation of correlated random fields is needed. One possible way is the Karhunen-Loève expansion (see [22]), which is basically a projection of a given random field onto the eigenvector basis, ortho-normal in $L_2(\mathcal{D})$, obtained by the Fredholm eigenvalue problem of the second kind (11),

$$\int_{\mathcal{D}} \text{Cov}_\gamma(x, y) \Phi_i(x) dx = \rho_i \Phi_i(y), \quad y \in \mathcal{D} \quad (11)$$

The solution of this eigenvalue problem for any domain $\mathcal{D}$ is obtained using finite element techniques and yields to a way to synthesize the two Gaussian random fields $\gamma_{i=1,2}$,

$$\gamma(x, y) = \sum_{i=1}^{\infty} \sqrt{\rho_i} \Phi_i(x) \xi_i(\omega) \quad (12)$$

and consequently σ_y and e_f through relations (9). In Eq. 12, the $\xi_i(\omega)$ are uncorrelated Gaussian random variables (with unit variance and zero mean) and thus independent. Having a computational approach in mind, the infinite sum (12) must be truncated. Figure 11 shows the covariance function synthesized using a different number of modes in KL expansion. These computations have been performed using a modified version of the finite element code FEAP (see [39]).

5.2 Size Effect and Correlation Length

As mentioned before, the 1D macro-model we consider here is based on a strong discontinuity model (see [14]) which leads to the possibility to couple diffuse plasticity or damage (describing the volumetric dissipation due to the homogeneous

micro-cracking which takes place in the FPZ) with surface dissipation at the macro cracks. The latest drives the stress to zero without any mesh dependency. Moreover no special precaution in order to take account for size effect have been taken except considering lognormal correlated random fields for the macroscopic quantities σ_y and e_f .

Considering tensile tests, three different lengths have been treated under displacement control (0.01, 0.1 and 1 m truss), keeping the correlation length equal to $L_c = 0.01$ m. These three cases will be called respectively small, medium and large. It is worth to note that for the medium case, the bar is the same size as the correlation length.

Computing a realization (evaluation of the random field at a given point in the stochastic domain) of γ_1 and γ_2 via the truncated Karhunen-Loève expansion on each of these bars as described previously and using the relations (9) lead to an efficient computational way of representing each realization the fields σ_y and e_f . The normalized fluctuating part of one realization of σ_y and e_f for each of these bar is shown on Fig. 12. It is worth to note that the larger the bar is with respect to the correlation length L_c the more fluctuating the random fields are. Thus, the more these macroscopic properties are likely to have lower bounds. In terms of strength, such lower bounds obviously lead to a weaker behavior.

Each of these independent realizations is used to perform Monte Carlo with 10,000 integration points for each case using the coFeap CTL component. Figure 13 presents the cumulative density functions for the maximum load. Considering a given percentile of broken bars, it is worth to note that the smaller the bar is, the bigger is its ultimate stress, respectively 3.68 MPa for the small truss, to 3.3 MPa for the medium one and to 2.87 MPa for the large one. In other words, the strength of the structure is directly linked to its size. The larger is the structure comparing to the correlation length, the weaker is the structure. Hence, this stochastic way of modelling quasi-brittle failure naturally reveals size effect. Figure 13 shows the 99% confidence interval with respect to the computations that have been made. For each bar, none of these error bars are overlapping. This point leads to the conclusion that the number of stochastic integration points used for the Monte Carlo process is large enough and thus that the results are accurate.

Clearly, the correlation length here plays the key role. Comparable to the characteristic length which appears in the non-local theory, it can be linked to the size of the Fracture Process Zone (FPZ) where micro-cracking occurs. The more the size of the FPZ prevails on the global size of the structure (which is the case for the small bar), the more similar to Continuum Damage Mechanics (CDM) the structure's behavior is. On the opposite, if the FPZ size is negligible with respect to the size of the structure (i.e. the large bar), its influence on the global behavior of the structure is small. Thus macro-crack occurs following Linear Fracture Mechanics (LFM). Modelling the behavior of quasi-brittle materials is an attempt to link these two limit behaviors (LFM and CDM). Figure 13 shows that one possible way is to consider correlated random fields to describe macroscopic quantities.

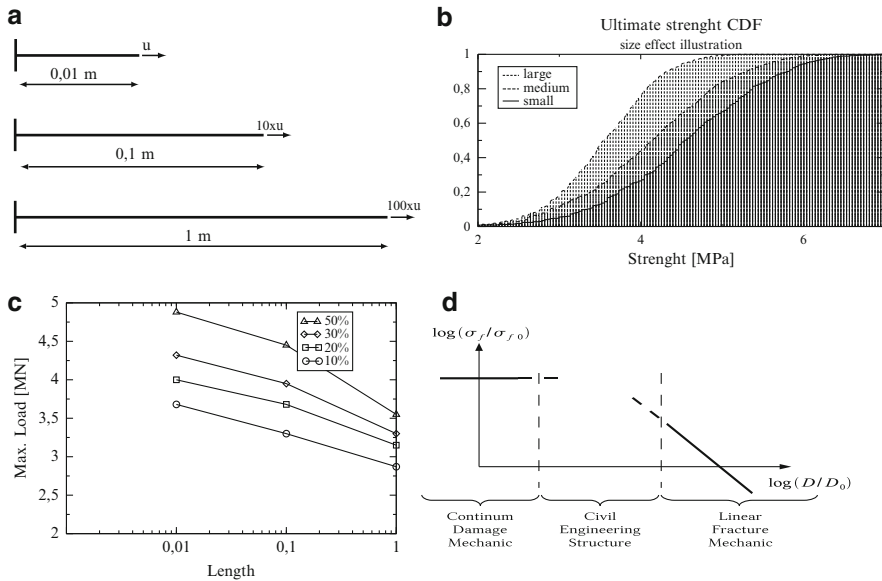


Fig. 13 (a) Short, medium and long bar, (b) Ultimate stress cumulative distribution (c) Probabilistic size effect diagram, (d) Standard size effect illustration

6 Concluding Remarks

When dealing with Finite Elements modelling of civil engineering structures built of heterogeneous materials, we can develop very robust phenomenological macro-models adapted to large structures computations. To achieve robustness, such models are based on homogeneous material representation of real material heterogeneities, introducing the key concept of RVE, to be determined depending on the chosen property and a prescribed relative error. Even though these macro-models can work well for a particular loading program, they lack the predictive value which is mainly due to their inability to correctly represent failure mechanisms at finer scales, neither their origin nor (even less) their evolution.

In order to improve predictive modelling of failure, the meso-scale has been here chosen as the one being pertinent to describe failure mechanisms. At this scale, cement-based composite materials (concrete, mortar etc.) are properly interpreted as heterogeneous and a special structured mesh methodology has been developed. Any such structured mesh relies on a regular grid where elements are not constrained to the physical interfaces between the different phases and can also contain phase interface. For any element crossed by phase interface, as shown here in detail for the classical CST elements, we explained how to enhance the elements kinematics accordingly by using the Incompatible Modes Method providing two kind of discontinuities. The first one consists in a strain discontinuity inside the element in order to model different elastic properties of the two phases. The second discontinuity

corresponds to a displacement one and allows to model the interface failure (e.g. debonding) along with the two different mechanisms, normal and tangential. We also showed that no remeshing is needed with such two-phases elements and structured mesh strategy.

With such an efficient failure modelling tool in hand we presented how to take into account for the variability of the geometrical description of a heterogeneous material at the meso-scale level. Considering cement-based materials, macroscopic properties such as mechanical strength are known to be strongly dependent from the porosity (voids volume fraction). Hence meso-scale geometry has been modelled by using modified Gibbs points processes with circular voids inside a perfectly-plastic Drucker-Prager matrix. Although the material properties of the two phases are assumed to be deterministic, this variability leads to a stochastic problem to be solved. In this work we employed the classical Monte-Carlo method in order to produce the statistical moments of the desired quantities. Using the Components Template Library (CTL) and the Finite Elements code FEAP we produced 10,000 independent realizations leading to first and second order statistics of macroscopic properties like elastic modulus, yield stress and ultimate stress. Moreover we determined the 5% relative error morphological RVE for different voids volume fractions.

Finally, based on the macroscopic properties (e.g. yield stress) covariance function (namely the covariance length), we showed that the proposed macro-model might provide a straight-forward modelling for size effect. Such size effects are a major issue in modelling quasi-brittle failure like cement-based one and bridge two limit cases, Continuum Damage Mechanic and Linear Fracture Mechanics as in Weibull's theory. In between, several authors have proposed size effects laws corresponding to different kind of structures and loading paths, or tried to model this particular feature. One attempt consists in using nonlocal models and retrieve size effects through their characteristic lengths, although this length has no physical basis. Here, we showed that the use of macroscopic correlated random fields naturally leads to size effects. Namely, the correlation length appearing in the random fields covariance functions is acting like a length scale. Another key ingredient of our development pertains to a phenomenological model that can account for both the fracture process zone (FPZ) and the localized failure introducing the displacement discontinuity and softening. This particular feature leads to the possibility to retrieve the size effect governed response that remains valid anywhere between the two limit cases: the one described Continuum Damage Mechanics, where the FPZ is dominant failure mechanism, and another defined by Linear Fracture Mechanics, where the displacement discontinuity quickly takes over failure leaving a negligible FPZ. This method might be viewed as an extension of Weibull's theory which can be retrieved considering uncorrelated random field.

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References

1. Bathe, K.J.: *Finite Element Procedures*. Prentice Hall, Upper Saddle River (1996)
2. Bažant, Z.P.: Probability distribution of energetic-statistical size effect in quasibrittle fracture. *Prob. Eng. Mech.* **19**, 307–319 (2004)
3. Bornert, M., Bretheau T., Gilormini, P.: *Homogeneisation en mécanique des matériaux*, Hermes-Science, Paris (2001)
4. Brancherie, D., Ibrahimbegovic, A.: Novel anisotropic continuum-damage model representing localized failure. Part I: Theoretical formulation and numerical implementation. *Eng. Comput.* **26**(1–2), 100–127 (2009)
5. Caflisch, C.A.: Monte Carlo and quasi-Monte Carlo Methods. *Acta Numerica* **7**, 1–49 (1998)
6. Carpinteri, A.: On the mechanics of quasi-brittle materials with a fractal microstructure. *Eng. Frac. Mech.* **70**, 2321–2349 (2003)
7. Colliat, J.B., Hautefeuille, M., Ibrahimbegovic, A., Matthies, H.: Stochastic approach to quasi-brittle failure and size effect. *Comptes Rend. Acad. Sci.: Mech.* **335**, 430–435 (2007)
8. Dolarevic, S., Ibrahimbegovic, A.: A modified three-surface elasto-plastic cap model and its numerical implementation. *Comput. Struct.* **85**, 419–430 (2007)
9. Gilormini, P.: A shortcoming of the classical non-linear extension of the self-consistent model. *C.R. Acad. Sci. Paris* **320** (116), 115–122 (1995)
10. Hautefeuille, M., Melnyk, S., Colliat, J.-B., Ibrahimbegovic, A.: Failure model for heterogeneous structures using structured meshes and accounting for probability aspects. *Eng. Comput.* **26**(1–2), 166–184 (2009)
11. Ibrahimbegovic, A.: *Nonlinear Solid Mechanics: Theoretical Formulations and Finite Element Solution Methods*, Springer, Berlin (2009)
12. Ibrahimbegovic, A., Brancherie, D.: Combined hardening and softening constitutive model of plasticity: precursor to shear slip line failure. *Comp. Mech.* **31**, 88–100 (2003)
13. Ibrahimbegovic, A., Markovic, D.: Strong coupling methods in multiphase and multiscale modeling of inelastic behavior of heterogeneous structures. *Comp. Method Appl. Mech. Eng.* **192**, 3089–3107 (2003)
14. Ibrahimbegovic, A., Melnyk, S.: Embedded discontinuity finite element method for modeling of localized failure in heterogeneous materials with structured mesh: an alternative to extended finite element method. *Comp. Mech.* **40**, 149–155 (2007)
15. Ibrahimbegovic, A., Wilson, E.L.: A modified method of incompatible modes. *Commun. Appl. Numer. Meth.* **7**, 187–194 (1991)
16. Jaynes, E.T.: *Probability Theory: The Logic of a Science*. Cambridge University Press, Cambridge (2003)
17. Kanit, T., Forest, S., Galliet, I., Mounoury V., Jeulin, D.: Determination of the size of the representative volume element for random composites: statistical and numerical approach, *Int. J. Solid Struct.* **40**, 3647–3679 (2003)
18. Kassiotis, C., Hautefeuille, M.: *coFeap's Manual*, LMT-Cachan Internal Report, **2**, (2008)
19. Kendall, M.G.: On the reconciliation of theories of probability. *Biometrika* **36**, 101–116 (1949)
20. Kucerova, A., Brancherie, D., Ibrahimbegovic, A., Zeman, J., Bitnar, Z.: Novel anisotropic continuum-damage model representing localized failure. Part II: Identification from tests under heterogeneous test fields. *Eng. Comput.* **26**(1–2), 128–144 (2009)
21. Ladeveze, P., Loiseau O., Dureisseix, D.: A micro-macro and parallel computational strategy for highly heterogeneous structures. *Int. J. Numer. Meth. Eng.* (2001)
22. Loève, M.: *Probability Theory* – 4th edn, **1** Macmillan, New York, NY (1977)
23. Markovic, D., Ibrahimbegovic, A.: On micro-macro interface conditions for micro-scale based fem for inelastic behavior of heterogeneous material. *Comput. Meth. Appl. Mech. Eng.* **193**, 5503–5523 (2004)
24. Matthies, H.G., Niekamp, R., Steindorf, J.: Algorithms for strong coupling procedures. *Comput. Meth. Appl. Mech. Eng.* **195**(17–18), 2028–2049 (2006)
25. Niekamp, R., Matthies, H.G.: *CTL: A C++ Communication Template Library*, GAMM Jahreshauptversammlung in Dresden, 21. 27. March (2004)

26. Niekamp, R., Markovic, D., Ibrahimbegović, A., Matthies, H. G., Taylor, R. L.: Multi-scale Modelling of Heterogeneous structures with inelastic constitutive behaviour: Part II – Software coupling implementation aspects. *Eng. Comput.* **26**, 6–28 (2009)
27. Ostoja-Starzewski, M.: Material spatial randomness: From statistical to representative volume element. *Probabilistic Eng. Mech.* **21**, 112–132 (2006)
28. Pijaudier-Cabot, G., Bažant, Z. P.: Nonlocal damage theory. *J. Eng. Mech.* **113**, 1512–1533 (1987)
29. Sab, K., Lalaai, I.: Une approche unifiée des effets d'échelle dans les matériaux quasi fragiles, *C. R. Acad. Sci. Paris II* **316**(9), 1187–1192 (1993)
30. Shannon, C.E.: A mathematical theory of communication. *Bell Syst. Tech. J.* **27**, 379–423 and 623–656, J (1948).
31. Simo, J.C., Taylor, R.L.: Consistent tangent operators for rate-independent elastoplasticity. *Comput. Meth. Appl. Mech. Eng.* **48**, 101–118 (1985)
32. Simo, J.C., Oliver, J., Armero, F.: An analysis of strong discontinuity induced by strain softening solution in rate independent solids. *Comp. Mech.* **12**, 277–296 (1993)
33. Smolyak, S.A.: Quadrature and interpolation formulas for tensor products of certain classes of function. *Soviet Math. Dokl.* **4**, 240–243 (1963)
34. Soize, C.: Maximum entropy approach for modeling random uncertainties in transient elastodynamics. *J. Acoust. Soc. Am.* **109**(5), 1979–1996 (2001)
35. Weibull, W.: A statistical distribution function of wide applicability. *J. Appl. Mech.* 293–297 (1951)
36. Wilson, E.L.: The static condensation algorithm. *Int. J. Num. Method Eng.* **8**, 199–203 (1974)
37. Wilson, E.L., Taylor, R.L., Doherty, W.P., Ghaboussi, J.: Incompatible displacement models, *Numerical and Computer Methods in Structural Mechanics*, pp. 43–57, Academic, New York (1973)
38. Yaman, I.O., Hearn, N., Aktan, H.M.: Active and non-active porosity in concrete. Part I: Experimental evidence. *Mater. Struct.* **15**, 102–109 (2002)
39. Zienkiewicz, O.C., Taylor, R.L.: *The Finite Element Method*, 6th edition, Vols. 1 and 2, Elsevier, Oxford (2005)

The Scatter of Eigenfrequencies in Beams Made of Metal Foam

Daniel Schwarzer and Carsten Proppe

Abstract Due to their useful properties, metal foams became an interesting, often utilized and investigated material. Recent applications are in areas, where dynamic processes play a significant role. In the huge amount of literature about metal foam, mainly the material properties like strength and stability are investigated but the dynamic behavior is rarely the object of research. Therefore this work investigates the principal vibration behavior of heterogeneous metal foam by examining the eigenfrequencies of bending beams in dependence on the irregular microstructure of the foam.

First of all the linear elastic properties of metal foam and their variations have to be investigated. Therefore, mesoscopic three dimensional stochastic volume elements are sampled including the effects of inhomogeneities like varying thickness along a ligament, pre-deformed ligaments, imperfections, partially closed cell faces or non-planar cell faces. In order to perform the step from the meso- to the macroscale, the mechanical properties are expressed as normally distributed random fields with a determined autocorrelation function or the power spectral density.

These random fields are used to predict the eigenfrequencies of Timoshenko beams made of metal foam. Therefore the Karhunen-Loève expansion and the Spectral Representation are derived analytically for the determined data of metal foam and used as two realization generators in Monte-Carlo-Simulations. The results are compared with experiments.

1 Introduction

In the last years metal foams became a frequently employed material. This development is due to the reason that metal foam can be used in mainly three special applications: Since metal foam has a relatively high stiffness compared to its mass,

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it can be utilized as energy absorbing elements in crash mechanics. Some modern cars use side skirts made of foamed metal. The advantage of this property is also taken in dynamic tasks, where structures are often accelerated and decelerated on a large scale. Here the mass has to be reduced without a loss of stiffness. One application of this are the carriages in modern machine tools [12]. The second property, why metal foam became more and more interesting, is its capability to act as an isolator for vibrations. Therefore chucks of modern machine tools contain a layer of metal foam in order to damp the vibrations coming from the cutting process [12]. This can also be used in acoustic applications. The third argument for metal foam is the huge surface in a small volume, which is needed as a basis bearing in catalytic converters or in filters. All these applications have in common that a knowledge of the mechanical properties, especially of the properties in vibration tasks, must be developed.

On the other hand, foams in general have the problem that their properties show – due to their irregular structure – a huge scatter yielding an uncertainty: It is quite important to know if a side skirt of car absorbs in a crash the amount of energy at a certain stress level or not. Thus also the scatter of the properties has to be investigated.

Since the length scale of the microstructure of metal foam is on the order of magnitude of the real problems to be examined (this scale is called from now on mesoscale), homogeneous material theories can not be applied. Thus, the irregular structure has to be taken into account. As a result of the different manufacturing processes for foams [1], the mesoscopic structure varies [9]: Foams can consist of open or closed cells and the shape of the structure and therewith the lengths of the ligaments and the connectivity at the vertices vary in a huge manner. Further on, there exist mesoscopic inhomogeneities like imperfections, pre-deformed ligaments, non-planar cell faces and others. All these points have an influence on the material properties and therewith on the behavior of metal foam parts.

This paper is split into two parts: In the first part the linear elastic properties are investigated with the help of a mesoscopic model containing structural differences like different cross section shape or varying thickness along a ligament and inhomogeneities like pre-deformed ligaments, imperfections, partially closed cell faces or non-planar cell faces. With the help of this model, the distributions of linear elastic properties of Aluminium foam are predicted. Afterwards the correlation function and other statistic characteristics for these properties, like the power spectral density, are calculated in order to describe the properties as random fields. In the second part of this paper the eigenfrequencies of beams made of metal foam and their scatter are predicted with the help of Monte-Carlo-Simulations. For these Monte-Carlo-Simulations different generators of the realizations, the Karhunen-Loève expansion and the Spectral Representation, are compared and applied to the investigated problem. Finally the results for the eigenfrequencies are compared to the results of experiments.

The paper closes with some concluding remarks.

2 Determination of the Linear Elastic Properties

In order to calculate the linear elastic properties of metal foam, mesoscopic volume elements are created randomly and loaded by boundary conditions yielding an upper (kinematic uniform boundary conditions KUBC) and a lower bound (static uniform boundary conditions SUBC) for the elastic properties. This procedure is often used in literature but mainly for the prediction of the volume size of a statistically representative volume element (RVE) used in homogenization schemes (for example [15, 16]). The sample size, for which these two bounds converge to the same value, defines the size of the RVE. Therefore the mechanical properties of the RVE are theoretically deterministic – in the sense of being accurate enough to represent mean constitutive response [5].

On the other hand, these homogenization techniques are based on the condition that the scale of the microstructure and the scale of the observed mechanical components can be separated thanks to the huge difference in their characteristic lengths. Unfortunately, the characteristic length scale of metal foam is in many applications not much smaller than the characteristic length scale of the component to be investigated. For these reasons, no RVE can be defined and the homogenization schemes cannot be applied. One has to use stochastic volume elements (SVE) instead. In this case the above mentioned method of loading different boundary conditions can still be adopted yielding so-called apparent properties [13], which are still the bounds of the effective properties of interest. For the stiffness tensor $\mathbf{C}$ and the compliance tensor $\mathbf{S}$, the following condition holds:

$$(\mathbf{S}^{\text{Reuss}})^{-1} \leq (\mathbf{S}_{\text{apparent}}^{\text{SUBC}})^{-1} \leq \mathbf{C}_{\text{effective}} \leq \mathbf{C}_{\text{apparent}}^{\text{KUBC}} \leq \mathbf{C}^{\text{Voigt}},$$

where the upper and lower bounds are bounded themselves by the classical bounds of Voigt and Reuss.

Since the mechanical properties of metal foam are estimated with the help of SVEs, not only their mean value but also their scatter can be calculated. Also the influence of mesoscopic inhomogeneities on the mechanical behavior can be measured and evaluated. This is done in the following subsections.

2.1 Generation of Structure

Since the basic deformation mechanism of metal foams is bending dominated, the mesoscopic model is based on the idea of Gibson and Ashby [9] that foam consists of a network of connected Timoshenko type beams with the material properties of the solid structure and circular cross sections. These three dimensional networks are generated either in a regular manner with the help of Kelvin cells or irregularly by randomly perturbing the vertices positions of the Kelvin cells or with the help of

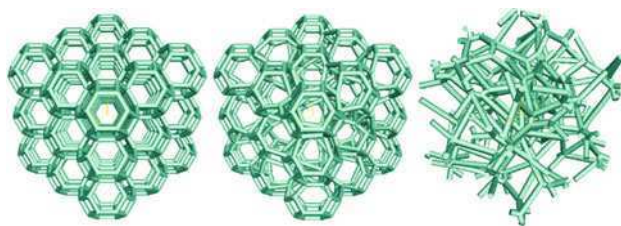


Fig. 1 Regular and irregular (misaligned and Voronoi) structures

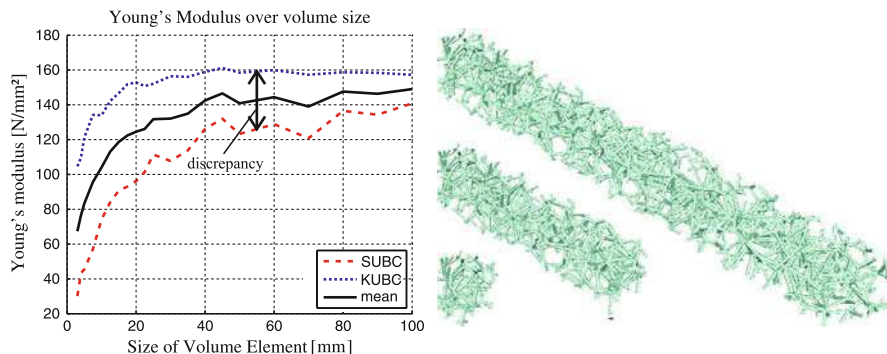


Fig. 2 Size effect of Young's modulus

cells generated by a Voronoi tessellation (Fig. 1). The latter method is based on the growth of grains from randomly chosen but uniformly distributed points. The input parameters to these generators are the number of cells in the SVE, the material properties of the solid and the relative density, which is the density of the foam divided by the density of the solid. The relative density is the most important parameter for metal foams, since other quantities like for example the constant cross sections area, can be calculated from it and since there are some formulas for basic estimations in dependence on this parameter [9].

In order to predict the elastic material constants in each direction, each realization of the ligament network is loaded by different load cases of the boundary conditions mentioned above and solved with the help of the finite element method. Afterwards the apparent material parameters (here mainly Young's modulus in three directions) are calculated from the results. For each parameter set to be investigated, at least 50 realizations are simulated and the mean values and standard deviations of these realizations are calculated. The calculations are done by generating the SVE in MatLab® and solving the gained equations with the help of the commercial FE solver Abaqus®. The statistical evaluation is done again in MatLab®.

In order to study the principal behavior of the SVE, macroscopic beamlike structures are considered and generated from a Voronoi tessellation. Starting from one end of these structures, smaller volume elements with increasing size in one

dimension are cut out and analyzed. From this, the Young's modulus can be plotted in dependence on the dimension of the volume element in the direction of the macroscopic beam's length axis.

As can be seen in Fig. 2, the upper and lower bound of the Young's modulus increase with increasing sample size until a certain size of the volume element (here about 40 mm) is reached. This behavior is due to a size effect of the model. For volumes larger than this size, the values remain approximately at the same level and only the difference between the two bounds becomes smaller. This means that the size of the volume element must be large enough to suppress the size effect.

From similar calculations with cubic volume elements the same results can be obtained. Additionally the shape of the compliance tensor $\mathbf{S}$ can be investigated. For this and from the calculation, $\mathbf{S}$ is written in the notation of Federov and Cowin

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \sqrt{2}\varepsilon_{23} \\ \sqrt{2}\varepsilon_{13} \\ \sqrt{2}\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & 0 & 0 & 0 \\ S_{2211} & S_{2222} & S_{2233} & 0 & 0 & 0 \\ S_{3311} & S_{3322} & S_{3333} & 0 & 0 & 0 \\ & & & S_{2323} & 0 & 0 \\ \text{sym.} & & & & S_{1313} & 0 \\ & & & & & S_{1212} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\sigma_{23} \\ \sqrt{2}\sigma_{13} \\ \sqrt{2}\sigma_{12} \end{bmatrix} \quad (1)$$

meaning that the material fulfills, depending on the structure generator and the size of the volume element, one of the following symmetries [2, 3]: orthotropic, tetragonal, transversely isotropic, cubic or isotropic symmetry.

As can be seen from the graphical representations [2] of the generalized Hooke's law (Fig. 3), where the Young's modulus is plotted in all direction of the space, the Young's modulus of the regular Kelvin cell model has a spherical shape and is therefore isotropic. Small disturbances of these regular cells yield also for quite small volume elements isotropic behavior. Interestingly, the symmetry group of the Voronoi cell generator changes with increasing size of the volume element. For small volume elements no symmetry group can be determined, while for bigger sizes a cubic symmetry turns out. Increasing the sample size further on, the Voronoi cells show an isotropic behavior as well. The size of the volume element is then in the same dimension as it is needed to suppress the mentioned size effect. Therefore metal foam can be treated as macroscopically isotropic.

2.2 Modeling Different Foams

In order to calculate the linear elastic properties of different foams, some properties such as the cross section shape and area or the irregularity of the structure must be adjustable. Here, the influence of these different modelings will be shown.

close to the sections shape of an open-cell Aluminium foam (Duocel[®]), the last one in Fig. 4 (T–) is close to the shape of a polyurethane foam [14]. For comparability the relative density of the model is kept constant meaning a constant cross section area. From this condition the characteristic length t of each section is calculated. The change of the cross sections shape yields only a change of the foam stiffness, which is of the same size as the change of the area moment of inertia of the beams cross section itself and which is independent of the used type of network generator.

In general, foams can be classified in two classes of topology: open and closed cell foams. For Aluminium as the solid material, Duocel[®] is an example for an open cell, while Alporas[®] is a closed cell foam. The open cell foams can be modeled as described above. To calculate the answer of closed cell models all faces are closed with the help of standard triangular shell-elements. The influence of this will be discussed in the next section. Here it is just mentioned that the adjustments in the following introduced hold for both types of foam: open and closed cell foams.

In a next step the influence of the three network generators on the Young's modulus is studied. The Kelvin-cell network, which is a regular structure, shows only one deterministic value of the Young's modulus. Introducing disturbed vertices positions yield a weaker irregular structure with a standard deviation for Young's modulus of about 5%. The Voronoi-algorithm with uniform distributed grains and no further restrictions (Poisson- or Γ -Voronoi) predicts an even weaker structure with a standard deviation of 15% (Fig. 5).

Introducing now as a restriction a minimum distance between the midpoints of two adjacent cells in the Voronoi algorithm (δ -Voronoi), like it is commonly done in literature (e.g. [4]), the generated structure turns in an uniformer and stiffer version. The bigger this minimum distance is, the higher the Young's modulus will be, but with a nearly constant standard deviation. The limit of the increasing of the minimum distance is reached, when the grains for the Voronoi model have all the same distance from each other, which results in a structure with nearly the same Young's modulus as the Kelvin cell model. The Kelvin cell model has two characteristic distances between the midpoints (one diagonal and one horizontal) for a given number of grains and a given volume element size, while the Voronoi model has only one. This fact is responsible for the standard deviation of the Young's modulus for Voronoi model. The results can be seen in Fig. 5, which compares the

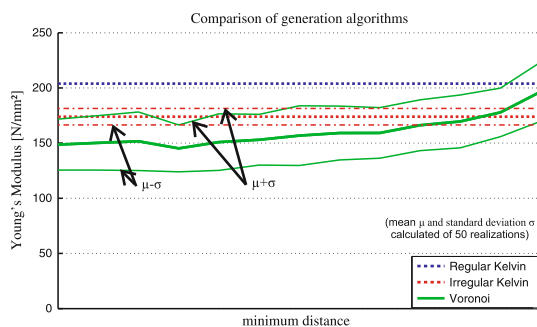
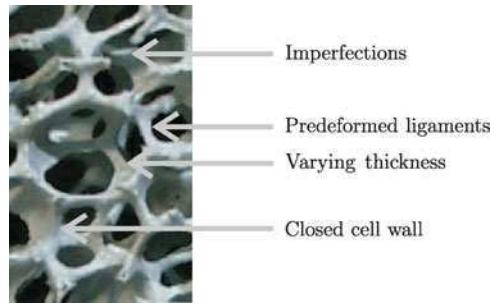


Fig. 5 Influence of a minimum distance on Young's modulus

Fig. 6 Overview of some mesoscopic inhomogeneities



obtained results with respect to the variation of the minimum distance between the midpoints of two adjacent cells. In the following the Voronoi algorithm is used with no minimum distance.

Figure 6 shows a picture of an open-cell metal foam. As can be seen, different mesoscopic inhomogeneities like varying ligament's thickness, pre-deformed ligaments, closed cell walls or imperfections may occur. In the following part of this paper, the influence of these inhomogeneities is investigated.

The first investigated mesoscopic inhomogeneity is the thickness of one ligament: It varies between the two vertices in such a way that there is more solid material close to the vertices than in the middle of the ligament. This is modeled by discretizing the ligament into several FEM-beam elements with different but constant thicknesses. The thickness of each (sub-) beam is calculated with the help of two mathematical approximations available in literature: Jang et al. [14] approximate the thickness of an open-cell Aluminium foam by fitting a polynomial of order four to measured cross section area data and calculate from that the thickness of the beams.

In contrast to that, Harders [11] models foam in two dimensions with rectangular cross sections shape and approximates the thickness by a simple quadratic function over the dimensionless length $\xi = \frac{x}{L}$ under the condition that the relative density remains constant. This yields two parameters: t_0 for the average thickness of the beam and thus for the relative density and t_{rel} for the design of the curvature of the ligament ($0 < t_{rel}$ because of minimum positive thickness in the middle of the beam, $t_{rel} \leq 1$ for realistic curvatures). Thus this approach can be fitted to different ligament shapes existing in different foams by adjusting the parameter t_{rel} . While extending this approach to three dimensions, it turns out that the calculated thickness is independent of the cross section shapes, which are examined in this work:

$$t = \sqrt{5}t_0 \left[4(1 - t_{rel}) \left(\xi - \frac{1}{2} \right)^2 - \frac{(1 - t_{rel})}{3} + \sqrt{\frac{1}{5} - \frac{4}{45}(1 - t_{rel})^2} \right]$$

Comparing the model with varying ligament's thickness to the one without, the model of Jang et al. predicts a higher value of the Young's modulus. The same effect

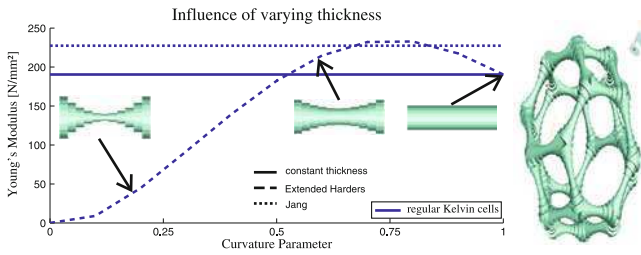


Fig. 7 Varying the cross sections area on the Young's modulus

can be achieved with the extended Harders' thickness for values of the curvature parameter t_{rel} between approx. 0.5 and 1. Therefore any real stiffness can be approximated through the derived formula by comparing the stiffness of only one beam. Decreasing the curvature parameter under 0.5 yields beams having such small cross sections in the middle that they are unable to carry much load and therefore the Young's modulus decreases.

The results are plotted for regular Kelvin cells in Fig. 7. For the other network generators the results are comparable: Only the mean value changes, while the scatter measured by the standard deviation remains at the same level. Even if the curvature parameter t_{rel} is randomly distributed in a small range, the level of the scatter remains the same.

2.3 Including Random Inhomogeneities on the Mesoscale

In this subsection the influence of different mesoscopic inhomogeneities like imperfections, some closed cells and predeformed ligaments or non-planar cell walls are investigated.

Figure 6 shows that some ligaments can be broken after the manufacturing process. These imperfections are modeled by randomly erasing nodes of the ligaments, which are discretized by more than one beam element. Since these ligaments do not contribute to the stiffness of the volume elements, they are erased and therefore the calculated relative density of the foam decreases. From this it follows that the calculated Young's modulus decreases as well. Already at a decrease of 15% of the relative density, the Young's modulus is reduced by 50%. Same results are obtained e.g. by Gan et al. [7]. The standard deviation of this model increases with more erased nodes. Also at 15% loss of the relative density, the influence of the irregular structure is no longer recognizable (Fig. 8).

As mentioned before, foams can be classified in open and closed cell foams. But even for open-cell foams, there are closed cells coming from the manufacturing process (Fig. 6). In order to capture the influence of these closed cells, some cells of the open-cell model are chosen randomly and closed by introducing triangular shell elements. The number of closed faces can be adjusted by a parameter which stands for the percentage of closed related to all cell faces.

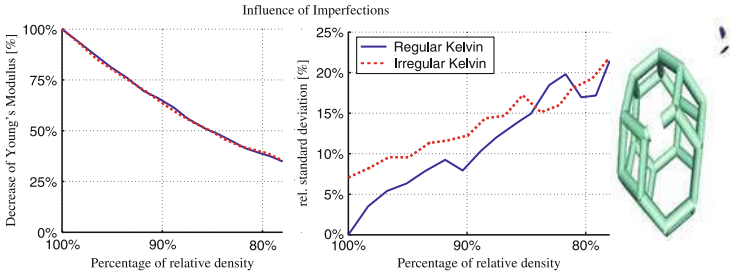


Fig. 8 Influence of imperfections on Young's modulus

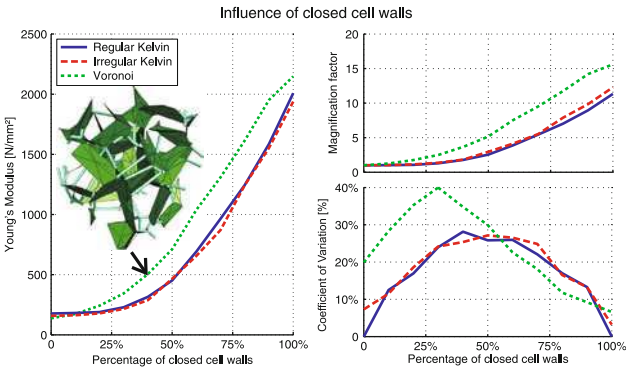


Fig. 9 Influence of closed cell faces on Young's modulus

In Fig. 9 the results are shown: for all three structure generators, the Young's modulus increases with a higher number of closed cells. But a percentage smaller than approx. 20% results in no difference of the Young's modulus. This means that some closed cells, coming from the manufacturing process, have no influence on the mean value of the Young's modulus, which is calculated for each percentage value out of 50 realizations. Increasing the percentage up to 100% yields a structure which is equivalent to a model for closed-cell foam. Hereby the Young's modulus is predicted approx. 11–12 times higher than the one for the open-cell foam for the Kelvin structures and about 15 times higher for the Voronoi structure (Fig. 9, top right).

Looking at the standard deviations occurring in these calculations (Fig. 9, down right), one can notice the influence of closed cells for the regular Kelvin cell model: Starting and ending at a deterministic model, the coefficient of variation increases quite fast and reaches its maximum value of approx. 30% at 50% closed cell faces. This result seems to be obvious. As in other investigations before, the disturbed vertex positions of the irregular Kelvin cells have nearly no influence. Only for models with less than 10% closed cells the influence can be noticed. In contrast to this behavior, the Voronoi model starts at a higher value for the standard deviation and

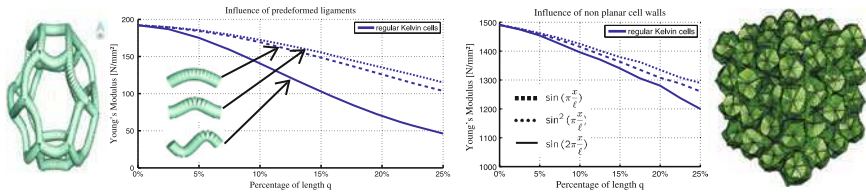


Fig. 10 Influence of predefined ligaments and non planar cell walls on Young's modulus

reaches its maximum already at about 30% of closed cell faces. It is also remarkable that the values for the coefficient of variation for closed cell foams are smaller than the one for open cell foam.

Until now the ligaments and the cell faces were straight or planar respectively. As can be seen in Fig. 6, this assumption does not hold in general. In order to model predefined ligaments, the ligaments are discretized again into several (sub-) beams. The connection points of these (sub-) beams are then moved in a direction perpendicular to the ligaments axis, which is calculated with the help of Kardan angles. Thereby the third Kardan angle is chosen at random. To determine the absolute value of the movement for each node, different shapes like sine, halfsine and halfsine squared are assumed depending only on one parameter: its amplitude. Since the ligaments have different length, the amplitude is chosen as a percentage q of its length. These three shape forms are chosen, because the first and the second represent the real ligament's shapes while the last has no rotation at the vertex' position and therefore allows to study the influence of the cross section rotation. To model the non-planar cell faces, the open cell model with predefined ligaments is taken and the cell faces are closed afterwards.

The influence of these two inhomogeneities is the same (see Fig. 10): All shape forms decrease the Young's modulus dramatically while increasing the parameter for the amplitude. For an amplitude of 25% of the ligaments length, the Young's modulus of the open cell foam is reduced by 75%, the one of the closed cell foam by 20% respectively. The influence is smaller for closed than for open cell foams. The same results can be seen comparing the coefficients of variation. As the difference between the results for the halfsine and the halfsine squared shape form is small, it can be concluded that the influence of the cross section rotation at the vertex is negligible when compared to the influence of the amplitude.

2.4 Validation

In order to validate the mesoscopic model for foam, one open cell (Duocel®) and one closed cell (Alporas®) aluminium foam is used. The necessary data of Duocel® is taken from the homepage of the manufacturer "ERG Materials and Aerospace Cooperation" [6]. Unfortunately, there are no data of the standard deviation of the Young's modulus available. The data of Alporas® were measured: For this purpose

Table 1 Comparison of Young’s modulus from experiments and of model

	Young’s modulus	
	Measured	Simulated
Duocel®	$\mu_E = 93.08 \text{ MPa}$	$\mu_E = 94.3 \text{ MPa}$
Alporas®	$\mu_E = 1,089 \text{ MPa}$	$\mu_E = 1,277 \text{ MPa}$
	$\sigma_E = 330 \text{ MPa}$	$\sigma_E = 285 \text{ MPa}$

the eigenfrequencies and eigenforms for traction, bending and torsion of 20 beams (2,000 mm length and a quadratic cross section area with an edge length of 50 mm) are measured. From these eigenfrequencies the Young’s and the shear modulus are calculated by assuming onedimensional vibrating structures (for example Timoshenko beams for the bending eigenfrequencies).

From these data, models are created: The model for Duocel® is made with an irregular Kelvin cell structure with 10 PPI (pores per Inch), a relative density of 8%, the varying thickness from Jang [14] and predeformed ligaments (amplitude equals 18% of length) with the cross section shape T+ from Fig. 4. The model for Alporas® is generated with the Voronoi structure generator with approx. 7.17 PPI, a relative density of 8,65%. It has non-planar closed cell faces.

From the results in Table 1, it can be seen that the Young’s modulus of both foams are predicted in the right order of magnitude. For Duocel®, the difference is less than 2%, for Alporas® less than 20%. The standard deviation are predicted for both models as too small, but the one coming from the Voronoi structure reflects the real standard deviation quite good. Both models include parameters, for example the amplitude of the predeformed ligaments or the number of broken ligaments, which are estimated by hand. More work has to be done to find values which are validated by experiments or CT-images. Anyhow, this presented model is good enough to be used in the prediction of the linear elastic material constants.

3 Statistical Evaluation

After building up the mesoscale model, investigating the influences of some inhomogeneities and validating with the help of experiments, the model can be used to predict the mean value and the scatter of the linear elastic properties of metal foam. The goal of this section is the description of the elastic properties as a random field in order to be used in further applications.

3.1 Determination of the Distribution Function

The basis of the calculation done in this subsection are about 1,000 analyzed realizations of a specific open and a specific closed cell foam each. From these calculations histograms of the upper and lower bound of the Young’s modulus are plotted and compared to analytical distribution functions. To find the optimal parameter for

the analytical functions, the maximum likelihood estimate is used. Afterwards the Kolmogorov-Smirnov test is carried out in order to find the best distribution. The result of the test is that the Young's modulus can be assumed as normally distributed for both bounds with the same standard deviation. Therefore, the pragmatic approach that the searched material parameters are the mean values of the two bounds can be used [10]. Thus, the Young's modulus can be assumed as normally distributed for both open- and closed-cell foams, even if physically non reasonable values (negative) may occur for the Young's modulus, since the Gaussian distribution is not bounded [17]. These values have to be cut of and the density function renormalized.

With the same technique the distribution of the mass density ρ , the shear modulus G and the bulk modulus K can be estimated: they can also be treated as normally distributed.

3.2 Determination of Correlations

In order to find the autocorrelation function for Young's modulus, 25 beam structures ($100 \times 10 \times 10$ mm) made of foam are generated with a Voronoi-network and analyzed by a method of moving cubes: Cubes of the same size are cut out of each of these beams at different positions along the beams length axis. For each cube the Young's modulus is calculated so that it is determined as a function of the position x on the beams length axis. This procedure is shown in Fig. 11. It is also repeated for different cube sizes and different resolution lengths d of the cubes position.

In order to calculate the autocorrelation data, the 25 obtained fields $E(x)$ are normalized assuming stationarity by

$$E_0(x) = \frac{E(x) - \mu_E}{\sigma_E}, \quad (2)$$

where μ_E and σ_E are the mean value and the standard deviation of the Young's modulus respectively. Afterwards for each field the autocorrelation data

$$\tilde{C}_E(\Delta) = \int_0^\ell E_0(x) E_0(x + \Delta) dx \quad (3)$$

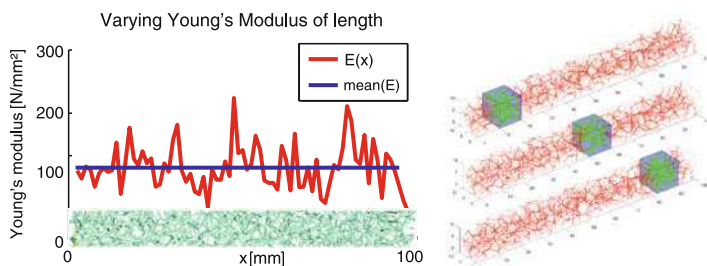


Fig. 11 Calculating the Young's modulus in dependence of position x

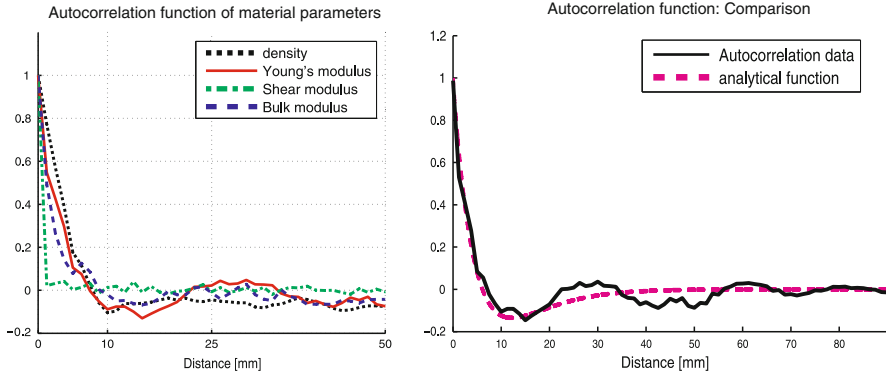


Fig. 12 Autocorrelation function: left for different material parameter, right comparison

is calculated as a function of the distance $\Delta = x_2 - x_1$, normalized by

$$\tilde{C}_{E0}(\Delta) = \frac{C_{EE}(\Delta)}{C_{EE}(\Delta = 0)} \quad (4)$$

and the mean value over all 25 fields is taken at each distance Δ . The results are shown in Fig. 12.

From these results it can be seen that the correlation data turns to zero when increasing the distance. This means that the random field can be treated as an ergodic random field, since the condition for ergodicity of a stationary Gaussian random field is $\lim_{\Delta \rightarrow \infty} \tilde{C}_E(\Delta) = 0$.

For the description of the autocorrelation data with an analytical function, the function

$$C_E^I(x) = e^{-c|x|} (u \cos w|x| - v \sin w|x|) \quad (5)$$

is used in a first step. In order to determine the parameters c, u, v, w , different conditions have to be fulfilled:

- (I) The autocorrelation function must be equal to 1 at the distance 0 ($C_E(x=0)=1$), since the Young's modulus at one position is strongly correlated with itself. This condition yields $u = 1$.
- (II) Since the data of the Young's modulus was normalized, the autocorrelation function must be mean free. Therefore the integral must be equal to zero:

$$\int_{x=-\infty}^{\infty} C_E(x) dx = 2 \int_{x=0}^{\infty} C_E(x) dx = 2 \frac{uc + vw}{c^2 + w^2} = 0. \quad (6)$$

Thus $v = -u \frac{c}{w}$.

With these conditions the suggested autocorrelation function is:

$$C_E(x) = e^{-c|x|} \left(\cos w|x| - \frac{c}{w} \sin w|x| \right). \quad (7)$$

Carrying out a least-square fit of this function to the given autocorrelation data, it turns out that $w \rightarrow 0$ and therefore the autocorrelation function degenerates to the one-parametric function

$$C_E^II(x) = e^{-c|x|} (1 - c|x|), \quad (8)$$

with the parameter $c = 0.164$. In Fig. 12, the autocorrelation data of the four material parameters, density ρ , Young's modulus E , shear modulus G and bulk modulus K , are compared to each other: The density, the Young's modulus and the bulk modulus behave in the same way so that their autocorrelation data is similar. Therefore the autocorrelation function can be fit by the same analytical function (8). In contrast, the shear modulus behaves in a different way and has to be investigated separately.

Cross correlations:

In the same manner the cross correlations between Young's modulus E , shear modulus G , bulk modulus K and the density ρ are determined, for example by

$$\tilde{C}_{E\rho}(\Delta) = \int_0^\ell E_0(x) \rho_0(x + \Delta) dx. \quad (9)$$

The cross correlation functions are shown in Fig. 13. The Young's modulus, the mass density and the bulk modulus are quite strongly correlated, while the shear modulus G is uncorrelated to the latter ones. Therefore the autocorrelation function for the density is assumed to be the same with that of Young's modulus.

3.3 Power Spectral Density

The power spectral density (PSD) of the material parameters can be calculated by a Fourier-Transformation of the autocorrelation function $C(x)$ (Wiener-Khintchine)

$$S(\omega) = \int_{x=-\infty}^{\infty} C(x) e^{-i\omega x} dx, \quad (10)$$

where ω is the angular space frequency with the unit [rad/mm]. For the suggested autocorrelation function with the sine and the cosine function C_E^I , the PSD results in

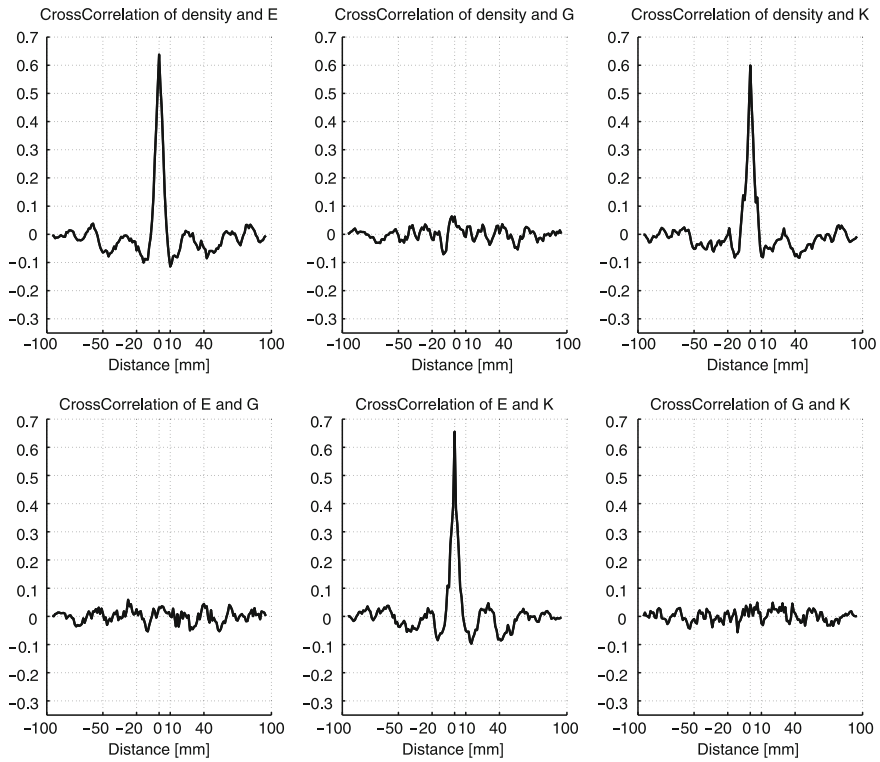


Fig. 13 Crosscorrelations between Young's modulus E , shear modulus G , bulk modulus K and the mass density ρ

$$\begin{aligned}
 S^I(\omega) &= \frac{2(uc - wv)\omega^2 + 2(uc^3 + ucw^2 + wvc^2 + w^3v)}{\omega^4 + 2(c^2 - w^2)\omega^2 + (c^4 + 2c^2w^2 + w^4)} \\
 &= \frac{p_1\omega^2 + p_0}{q_2\omega^4 + q_1\omega^2 + q_0},
 \end{aligned} \tag{11}$$

while for the degenerated autocorrelation function C^{II} the PSD is

$$S^{II}(\omega) = \frac{4c\omega^2}{(c^2 + \omega^2)^2}. \tag{12}$$

Since the PSD can also be calculated from the square value of the Fourier-Transform of the normalized and therewith mean-free material parameter over the position (for example Young's modulus $E_0(x)$), the PSD must be equal to zero at $\omega = 0$. For the first autocorrelation function this yields the condition

$$p_0 = uc^3 + ucw^2 + wvc^2 + w^3v = w(c^2 + w^2)(uc + vw) = 0, \tag{13}$$

which is the same condition like for zero mean value of the autocorrelation function in Eq. 6, since all parameter c, u, v, w are real and different from 0. In contrast, the degenerated autocorrelation function C'' fulfills this condition by definition, therefore $S''(\omega = 0) = 0$.

3.4 Conclusions of Statistical Evaluation

Until now, the results can be summarized as follows:

- The Young's modulus E , the density ρ , the shear modulus G and the bulk modulus K are normally distributed. Their mean μ and their standard deviation σ are known.
- The autocorrelation data of the Young's modulus E , the bulk modulus K and the density ρ is determined and expressed through an analytical function. Since the differences in their autocorrelation data are negligible and their cross correlation data shows a dependency on each other, the same parameter for this analytical function can be used. The associated PSD is known.
- As a consequence of the last two items, the Young's modulus E , the density ρ and the bulk modulus K can be written as normalized random fields $\alpha(x, \theta)$, where θ denotes an element of the set of outcome Θ of a complete probability space Ω_P .
- The shape of the autocorrelation data of the shear modulus G is different to the other material parameters. Also the cross correlations show that it is uncorrelated. Therefore the shear modulus has to be treated separately. Since the correlation data was examined using a Voronoi model with a moderate volume element size, this independence of the shear modulus G is in agreement with the cubic symmetry of the compliance tensor $\mathbf{S}$ e.g. in Fig. 3.

4 Application

In this section the previously obtained results are applied to the prediction of eigenfrequencies of metal foam structures. This information is important especially for those tasks, where vibrations play a significant role. Examples are the transmission behavior of foamed layers in drilling chucks, all sorts of vibrations in high accurate machine tools or in noise reduction tasks for the comfort in cars. The influence of metal foam is shown here with the help of simple, one dimensional problems: The calculation of bending eigenfrequencies of beams.

In order to predict the eigenfrequencies, two steps are necessary:

1. First of all, different realizations of the random fields for the Young's modulus $E(x, \theta)$, the shear modulus $G(x, \theta)$ and the density $\rho(x, \theta)$ have to be computed. This is done in literature commonly in two different ways: With the

Karhunen-Loève Expansion (KLE), starting with a spectral decomposition of the autocorrelation data and expanding the random field in a Fourier-type series with separated physical and stochastic domain, or with the Spectral Representation (SR) by a series of trigonometric functions with random phases.

2. Afterwards these realizations are used as input material data in Monte-Carlo FE simulations (MCS).

4.1 Generation of Realizations

For the calculation of realizations of a random field $\alpha(x, \theta)$ representing one of the parameter fields $E(x, \theta)$, $G(x, \theta)$ or $\rho(x, \theta)$, different possible ways exist in literature. In this work the commonly used ways, the KLE and the SR, are used and derived in the following sections for the current random fields. Other ways, like for example a linear filter of white noise as a point process, are not utilized since they have the disadvantage that they do not allow further analytical or semi-analytical investigations like stochastic FEM or a perturbation method.

4.1.1 Karhunen-Loève Expansion

With the help of the Karhunen-Loève expansion it is possible to separate the physical domain Ω and the probability space Ω_P through a series of orthogonal functions. These functions are the eigenfunctions of a Fredholm integral equation of second kind with the autocorrelation data as kernel function. Therefore the KLE is a spectral decomposition of the autocorrelation function [8]. With this, the realization of the random field can be written as

$$\alpha(x, \theta) = \sum_{n=0}^{\infty} \sqrt{\lambda_n} f_n(x) z_n(\theta), \quad (14)$$

where $z_n(\theta)$ are uncorrelated Gaussian random variables, since $\alpha(x, \theta)$ is a Gaussian field. Under a suitable assumption for the correlation function, the KLE is then almost surely convergent and one may truncate the series for $n > N$:

$$\alpha(x, \theta) \approx \sum_{n=0}^N \sqrt{\lambda_n} f_n(x) z_n(\theta) = \mathbf{z}_i(\theta) \mathbf{F}_i(x), \quad (15)$$

with $\mathbf{z}(\theta) = [z_1(\theta), z_2(\theta), \dots, z_N(\theta)]$ and $\mathbf{F}(x) = [\sqrt{\lambda_1} f_1(x), \sqrt{\lambda_2} f_2(x), \dots, \sqrt{\lambda_N} f_N(x)]^T$. The necessary eigenvalues λ_n and eigenfunctions $f_n(x)$ have to fulfill the Fredholm integral equation

$$\int_{x_1} C(x_1, x_2) f_n(x_1) dx_1 = \lambda_n f_n(x_2). \quad (16)$$

In addition the eigenfunctions $f_n(x)$ are normalized according to

$$\int_x f_n(x) f_m(x) dx = \delta_{nm}, \quad (17)$$

where δ_{nm} is the Kronecker delta function. In order to determine λ_n and $f_n(x)$, the integral $\int_a^b \dots dx_1$ in Eq. 16 is split into the two integrals $\int_a^{x_2} \dots dx_1 + \int_{x_2}^b \dots dx_1$, where a is the lower and b the upper boundary of the physical domain Ω respectively. Differentiating this equation four times with respect to x_2 and introducing the lower differentiations into this fourth differentiation yield a ordinary differential equation for $f_n(x)$ for the autocorrelation function $C''(x)$ given in Eq. 8

$$\lambda f_n''''(x) + c(4 - 2c\lambda) f_n''(x) + c^4 \lambda f_n(x) = 0. \quad (18)$$

The real solution of the differential equation for $f_n(x)$ changes depending on the parameter c and the eigenvalue λ . If $c\lambda \leq 1$ the solution is

$$f_n^I(x) = c_1^I \cos v_1 x + c_2^I \sin v_1 x + c_3^I \cos v_2 x + c_4^I \sin v_2 x, \quad (19)$$

with $v_{1/2}^2 = -c^2 + 2\frac{c}{\lambda}(1 \pm \sqrt{1 - c\lambda})$, while in the other case ($c\lambda > 1$) the solution degenerates to

$$f_n^{II}(x) = (c_1^{II} e^{-v_3 x} + c_3^{II} e^{v_3 x}) \cos v_4 x + (c_2^{II} e^{-v_3 x} + c_4^{II} e^{v_3 x}) \sin v_4 x, \quad (20)$$

with $v_3^2 = \frac{c}{\lambda}(c\lambda - 1)$ and $v_4^2 = \frac{c}{\lambda}$.

The four necessary boundary conditions for the solution of this differential equation can be computed in the same way like the differential equation was derived, and expressed by introducing the boundary of the physical domain Ω at $x = a$ or $x = b$ respectively through the following equations

$$0 = \lambda f_n''(a) - 2c\lambda f_n'(a) + c(c\lambda + 4) f_n(a) \quad (21)$$

$$0 = \lambda f_n''(b) + 2c\lambda f_n'(b) - c(c\lambda + 4) f_n(b) \quad (22)$$

$$0 = \lambda f_n'''(a) + c(4 - 3c\lambda) f_n'(a) + 2c^3 \lambda f_n(a) \quad (23)$$

$$0 = \lambda f_n'''(b) + c(4 - 3c\lambda) f_n'(b) - 2c^3 \lambda f_n(b). \quad (24)$$

Introducing solutions $f_n^I(x)$ or $f_n^{II}(x)$ respectively into these boundary conditions yield a homogeneous linear system of equations for the constants c_i^I and c_i^{II} ($i = 1, 2, 3, 4$). The condition for non-trivial solutions of this system of equation results in the characteristic equation for λ : $g(c, a, b, \lambda) = 0$. The solutions λ_n as the roots of $g(c, a, b, \lambda)$ have to be determined numerically since the characteristic equation is highly nonlinear in λ . With the help of the normalization condition (Eq. 17), the constants c_i^I and c_i^{II} respectively and therewith the eigenfunctions $f_n^I(x)$ and $f_n^{II}(x)$ can be determined.

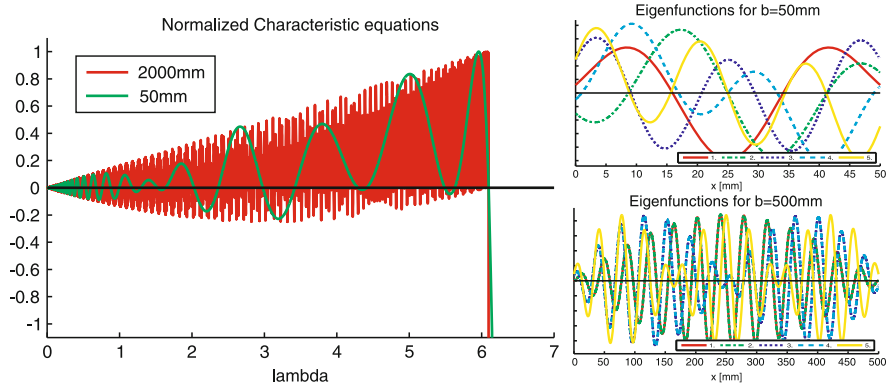


Fig. 14 KLE: Characteristic equations $g(c, a, b, \lambda) = 0$ for different beam lengths b ($a = 0$) and the first five eigenfunctions for $b = 50\text{ mm}$ and $b = 500\text{ mm}$

In Fig. 14 the characteristic equation is plotted for different beam lengths b ($a = 0$). Also the first associated eigenfunctions are shown. From these plots, it can be seen that (I) a longer beam has a characteristic equation with more significant roots λ , which have to be taken into account, and (II) a λ_n with a higher index n belongs to a eigenfunction $f_n(x)$ with a higher frequency ν_i ($i = 1, 2, 4$).

In order to determine the number of terms N in the expansion, the standard deviation σ_α of $\alpha(x, \theta)$ is calculated by

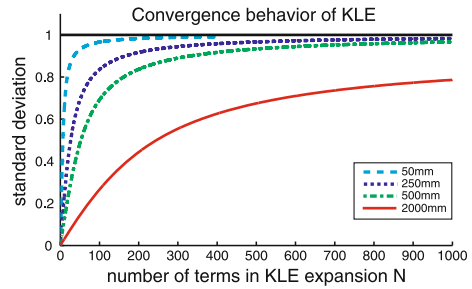
$$\begin{aligned} \sigma_\alpha &= \sqrt{\mathbf{E}[\alpha(x, \theta)^2]} = \sqrt{\sum_{n=0}^N \sum_{m=0}^N \sqrt{\lambda_n \lambda_m} f_n(x) f_m(x) \mathbf{E}[z_n(\theta) z_m(\theta)]} \\ &= \sqrt{\sum_{n=0}^N \lambda_n f_n^2(x)}. \end{aligned} \quad (25)$$

Since the random field $\alpha(x, \theta)$ is normalized, the standard deviation $\sigma_\alpha \rightarrow 1$ for $N \rightarrow \infty$. Therefore the standard deviation of the truncated sum is a measure for the quality of the KLE and it is plotted in Fig. 15 for different beam lengths over the number of terms N in the KL expansion.

From this figure, it can be seen that for longer beams more terms in the expansion have a significant influence. Since the ratio of the correlation length and the beam length turns to zero for longer beams, more and more terms belonging to eigenfunctions with higher frequencies have to be taken into account. The limit case of this tendency is that the autocorrelation function degenerates for huge beams to a function close to the Dirac-delta-distribution, whose spectrum includes all frequencies up to infinity. As mentioned before, the higher frequency eigenfunctions belong to eigenvalues λ_n with an increasing weight in the series expansion.

A KLE for the autocorrelation function C^I can be derived in a similar way [19].

Fig. 15 Convergence behavior of the KLE for different beam lengths



4.1.2 Spectral Representation

The spectral representation (SR) expands the random field $\alpha(x, \theta)$ as a sum of trigonometric functions within general random amplitudes and phases. Since this approach yields only weakly ergodic sample functions, the amplitudes are chosen to be deterministic. This change causes that the sampled functions are ergodic. The amplitudes are then only dependent on the PSD and the increment of the angular frequency [20, 21]. All together the SR reads

$$\alpha_0(x, \theta) = \sqrt{2} \sum_{n=0}^{N-1} A_n \cos(\omega_n x + \phi_n) = \sqrt{2} \sum_{n=0}^{N-1} \sqrt{2S(\omega_n)\Delta\omega} \cos(\omega_n x + \phi_n), \quad (26)$$

with the discrete angular frequencies $\omega_n = n\Delta\omega$, the frequency increment $\Delta\omega = \frac{\omega_c}{N}$ and the angular cut-off frequency ω_c , which defines the active region of the PSD. The phase angles are random variables uniformly distributed in the range $[0, 2\pi]$.

From the condition $S(\omega = 0) = 0$ (see Eq. 13) follows that the simulated field is periodic with the period length $\ell = \frac{2\pi}{\Delta\omega}$. In order to compute the SR over a beam with a given length, this length is set equal to the period length $\ell = b - a = \frac{2\pi}{\Delta\omega}$. The resolution Δx of the calculated points of the random field is $\Delta x \leq \frac{2\pi}{\omega_c}$, under the condition of avoiding aliasing. Therewith all necessary parameters can be expressed with the help of the two parameter cut off frequency ω_c and beam length ℓ . If a special resolution Δx is necessary for further calculations, the gained data points can be extrapolated linearly to the desired resolution.

As for the KLE, the standard deviation of the SR,

$$\sigma_\alpha = \sqrt{\mathbf{E}[\alpha(x, \theta)^2]} = \sqrt{\sum_{n=0}^{N-1} A_n^2} = \sqrt{\sum_{n=0}^{N-1} (2S(\omega_n)\Delta\omega)^2}, \quad (27)$$

can be taken to measure the quality of the representation. In Fig. 16 left, the influence of the cut off frequency $f_c = \frac{\omega_c}{2\pi}$ is shown for a beam with a length of 500 mm. In the right plot, the influence of the beam length for one constant angular cut-off frequency $\omega_c = 10$ rad/s is shown. From the above mentioned dependency of the

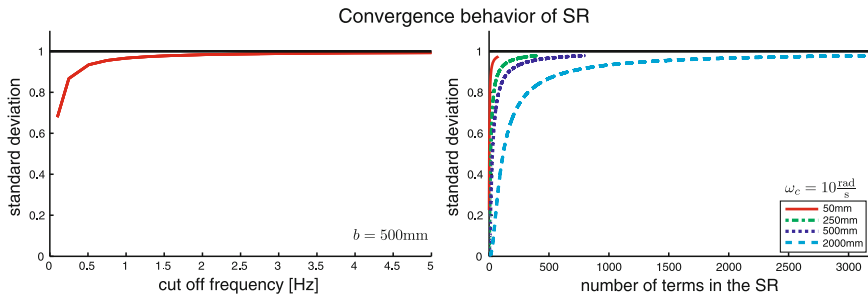


Fig. 16 Convergence behavior of the SR for different cut off frequencies and different beam lengths

number of terms $N = \frac{1}{2\pi}\omega_c \ell$ from the beam length and the cut off frequency, it can be seen that the computational effort increases fast for long beams with an adequate representation of the random field $\alpha(x, \theta)$.

4.2 Monte-Carlo Simulations

In this sub chapter, the two realization generators are used to predict the bending eigenfrequencies of beams. The beams are modeled with the FE software Comsol® since Comsol® has the opportunity to start with the weak formulation of the problem, discretize the physical domain with different types of shape functions and calculate the eigenfrequencies in dependence of over the physical domain varying material properties. Since Comsol® can be coupled with Matlab®, these realizations of the material properties are calculated with the help of Matlab® files, which are called by the solver of Comsol®. The solver itself can also be controlled by Matlab®, which allows for MCS. Here 1,000 realizations are calculated for each examined case.

The beams investigated in this work are modeled as Timoshenko beams with a square cross section of 50 mm side length and a shear correction factor of $\frac{5}{6}$. There-with all necessary geometric constants can be calculated. The beam is discretized by linear Lagrange shape functions with an element length of 0.25 mm. The integration is done using only on Gauss point in the middle of each element. In case of the SR, the data of the material properties is projected linearly to these positions.

The three necessary material properties for a linear elastic and dynamic task are the Young's modulus $E(x, \theta) = \mu_E + \sigma_E E_0(x, \theta)$, the shear modulus $G(x, \theta) = \mu_G + \sigma_G G_0(x, \theta)$ and the density $\rho(x, \theta) = \mu_\rho + \sigma_\rho \rho_0(x, \theta)$. They are modeled with the help of the normalized random fields $E_0(x, \theta)$, $G_0(x, \theta)$ and $\rho_0(x, \theta)$ generated with the KLE or the SR respectively. Since the density ρ and the Young's modulus E are highly correlated with each other, the same realization of the generators are utilized. In a first step also the normalized shear modulus G_0 is modeled in the same way like the normalized Young's modulus E_0 and the normalized

Table 2 Linear elastic material properties of the utilized aluminium foam

	Mean value μ_i	Standard deviation σ_i	Coefficient of variation
Young's modulus E	1,100 MPa	300 MPa	27.3%
Shear modulus G	423 MPa	150 MPa	35.5%
Density ρ	232 kg/m ³	48 kg/m ³	20.1%

Table 3 Results: comparison of calculated values and values of experiments

	Bending Eigenfrequencies						
Beam length	2,000 mm					500 mm	400 mm
Number	1.	2.	3.	4.	5.	1.	1.
Experiments							
Mean value [Hz]	26.80	75.49	147.89	241.30	359.92	407.14	648.54
Std. dev. [Hz]	1.79	4.45	7.93	13.57	19.97	14.32	45.20
Coef. of var. [%]	6.69	5.90	5.36	5.62	5.55	3.52	6.97
Homogeneous Material							
Euler-Bernoulli [Hz]	27.97	77.12	151.19	249.93	373.35	447.65	699.46
Timoshenko [Hz]	27.91	76.63	149.31	244.86	362.24	432.48	663.70
MCS with Karhunen-Loève expansion							
Mean value [Hz]	26.93	73.95	143.86	235.90	348.58	411.53	633.04
Std. dev. [Hz]	0.86	2.28	2.76	6.56	9.75	22.31	32.27
Coef. of var. [%]	3.20	3.08	1.92	2.78	2.80	5.42	5.10
MCS with Spectral representation							
Mean value [Hz]	26.81	73.47	142.92	234.52	346.26	414.04	634.24
Std. dev. [Hz]	1.44	3.44	3.68	8.73	9.49	30.49	21.71
Coef. of var. [%]	5.40	4.68	2.57	3.72	2.74	7.36	3.42

density ρ_0 . The chosen parameter for the KLE are the number of terms in the series $N = 1,000$ and for the SR a cut off frequency $\omega_c = 6$ rad/s. The utilized material data, measured from a closed cell aluminium foam Alporas[®], is summarized in Table 2.

The results of the Monte-Carlo Simulations are compared to the results of a modal analysis of beams with different lengths: 20 beams of 2,000 mm length, 40 beams of 500 mm length and 50 beams of 400 mm length. For the long beams the first five, for the shorter beams only the first bending eigenfrequencies were measured. The results of the experiments are summarized in Table 3 and they are compared to values of 1,000 MCS.

From these results the following conclusions can be drawn:

- Predicting the eigenfrequencies of heterogeneous material with the help of a homogeneous theory yield a mean value of the frequencies, which is about 5% higher than the real value. Using instead methods of stochastic mechanics reduces the predicted values to the same level or a little bit less than the real values. The difference between the homogeneous theory and the mean value of the experiments decreases with increasing sample size so that for huge beam

lengths the inhomogeneous material can be treated as homogeneous. These results can be verified with three dimensional FE simulations in Abaqus® using user-subroutines to vary only the Young's modulus over the position [18].

- The standard deviation and the coefficient of variation of the eigenfrequencies is predicted by the MCS in the same order of magnitude than the ones of the measurement. This means that the utilized methods (both: KLE and SR) work well in this application.
- Comparing the KLE with the SR shows that these two methods yield similar results for shorter beams. For longer beams the KLE predicts a smaller coefficient of variation. This is due to the representation of the random field through the KLE (see Fig. 15).

5 Conclusions

In this paper a procedure is introduced to predict eigenfrequencies of structures consisting of metal foam. The first step of the procedure is the calculation of the linear elastic properties depending on the irregularity of the beam network the foam consists of. The result is that the Young's modulus E , the shear modulus G and the density ρ can be treated as normally distributed. Also the influence of different mesoscopic inhomogeneities like varying thickness along the ligaments axis, pre-deformed ligaments, imperfections, closed cell faces and non-planar cell faces on these properties is shown. Further on, the autocorrelation function, the cross correlation data and the power spectral density of these material properties are determined. Thus, the linear elastic material properties can be expressed as normally distributed random fields.

Further on, two common ways for generating realizations of random fields, the Karhunen-Loève expansion and the Spectral Representation, are used in Monte-Carlo simulations of eigenvibration problems of bending beams of different length. The gained eigenfrequencies are compared to experimental data. From this comparison it can be concluded that the heterogeneity of the materials microstructure has an influence on the macroscopic properties: The mean values of the eigenfrequencies are smaller than the ones of beams with homogeneous material of similar mean material properties. Also, a scatter of the eigenfrequencies is observable. This effect has to be taken into account when using parts consisting of metal foam.

References

1. Banhart, J., Ashby, M.F., Fleck, N.A.: Cellular Metals and Metal Foaming Technology. MIT Publishing, Boston (2001)
2. Böhlke, T., Brüggemann, C.: Graphical Representation of the Generalized Hooke's Law. *Technische Mechanik* **21**, 145–158 (2001)

3. Chadwick, P., Vianello, M., Cowin, S.C.: A new proof that the number of linear elastic symmetries is eight. *J. Mech. Phys. Solids* **49**, 2471–2492 (2001)
4. Chen, C., Lu, T.J., Fleck, N.A.: Effects of imperfections on the yielding of two-dimensional foams. *J. Mech. Phys. Solids* **47**, 2235–2272 (1999)
5. Drugan, W.J., Willis, J.R.: A micromechanics-based nonlocal constitutive equation and estimates of representative volume element size for elastic composites. *J. Mech. Phys. Solids* **44**, 497–524 (1996)
6. <http://www.ergaerospace.com/foamproperties/aluminumproperties.htm>. Cited 19 Apr 2010
7. Gan, Y.X., Chen, C., Shen, Y.P.: Three-dimensional modeling of the mechanical property of linearly elastic open cell foams. *Int. J. Solids Struct.* **42**, 6628–6642 (2005)
8. Ghanem, R.G., Spanos, P.D.: *Stochastic Finite Elements: A Spectral Approach*. Springer, Hamburg (1991)
9. Gibson, L.J., Ashby, M.F.: *Cellular solids – Structure and properties*. Cambridge University Press, Cambridge (1997)
10. Gross, D., Seelig, T.: *Bruchmechanik mit einer Einführung in die Mikromechanik*. Springer, Berlin (2001)
11. Harders, H.: *Ermüdung von Aluminiumschaum*. Dissertation, Technische Universität Carolo-Wilhelmina zu Braunschweig (2005)
12. Hipke, T., Lange, G., Poss, R.: *Taschenbuch für Aluminiumschäume*. Aluminiumverlag, Düsseldorf (2007)
13. Huet, C.: Application of variational concepts to size effects in elastic heterogeneous bodies. *J. Mech. Phys. Solids* **38**, 813–841 (1990)
14. Jang, W.-Y., Kraynik, A.M., Kyriakides, S.: On the microstructure of open-cell foams and its effect on elastic properties. *Int. J. Solids Struct.* **45**, 1845–1875 (2008)
15. Kanit, T., Forest, S., Galliet, I., Mounoury, V., Jeulin, D.: Determination of the size of the representative volume element for random composites: statistical and numerical approach. *Int. J. Solids Struct.* **40**, 3647–3679 (2003)
16. Ostoj-Starzewski, M.: *Microstructural Randomness and Scaling in Mechanics of Materials*. Chapman & Hall/CRC, Boca Raton (2007)
17. Matthies, H.G., Brenner, C.E., Bucher, C.G., Soares, C.G.: Uncertainties in probabilistic numerical analysis of structures and solids Stochastic finite elements. *Struct. Saf.* **19**, 283–336 (1997)
18. Schwarzer, D., Proppe, C.: Macroscopic mechanical behavior of metal foam influenced by mesoscopic randomness. In: *Proceedings of the 10th International Conference on Structural Safety and reliability*, Osaka, Japan (2009)
19. Schwarzer, D., Proppe, C.: Stochastic description of the mechanical properties of metal foam obtained by mesoscopic modeling. In: *Proceedings of the 7th EUROMECH Solid Mechanics Conference* Lisbon, Portugal (2009)
20. Shinozuka, M., Deodatis, G.: Simulation of stochastic processes by spectral representation. *Appl. Mech. Rev.* **44**, 191–204 (1991)
21. Stefanou, G., Papadarakakis, M.: Assessment of spectral representation and Karhunen-Loève expansion methods for the simulation of Gaussian stochastic fields. *Comput. Method Appl. Mech. Eng.* **196**, 2465–2477 (2007)

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